PLURIHARMONICITY IN THE SHEAF OF COMPLEX LINES

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ABSTRACT

In this article, we prove an analog of Forelli’s Theorem for pluriharmonic functions, Theorem 1. Let D be a complete circular domain in $\mathbb{C}^n$ and the function $U, D \to \mathbb{R}$, defined in D, satisfies the following conditions:

1) $U$ is $M$-subharmonic somewhere, $U \in Msh \{0\}$

2) For each fixed $z \in D$ the function $U(\lambda z)$ is harmonic in the circle $\{ \lambda \in \mathbb{C}^n : \lambda z \in D \}$.

Then the function $U$ is pluriharmonic in D.

Keywords: M – sub harmonic functions, plurisubharmonic functions, holomorphic functions, harmonic functions, entire circular domain.

INTRODUCTION

Let’s begin with known definitions of circular domains.

Definition 1. Domain $D \subset \mathbb{C}^n$ is called entire circular if $\lambda z \in D$, for each $z \in D$ and $|\lambda| \leq 1$.

The results of this paragraph are directly related to the following Forelli’s theorem.

The theorem (Forelli). Let $D$ be an entire circular domain in $\mathbb{C}^n$. Let’s suppose that the function $U, D \to \mathbb{R}$ has the following features:

1) $U$ is infinitely continuously differentiable in the neighborhood of zero: $U \in C^\infty \{0\}$

2) For each fixed $z \in D$ the function $U(\lambda z)$ is harmonic in the circle $\{ \lambda \in \mathbb{C} : \lambda z \in D \}$.

Then, the function $U$ is pluriharmonic in D.

The example $K$ continuously differentiable function $U(z) = \frac{z_{k+1}^{k+1} \times \bar{z}_z}{|z|^2}$ $U(0) = 0$

show that the gained result is given incorrectly if the condition I - infinite smoothness is substituted with the existence of only a finite number of derivatives in the
neighborhood \(0 \in C^n\). We prove that the condition \(U \in C^\infty \{0\}\) may be replaced by the more convenient condition \(M – \text{subharmonic in some neighborhood of zero.}

Theorem 1. Let \(D\) be an entire circular domain in \(C^n\) and the function \(U, D \rightarrow R\) given in \(D\) satisfies the following conditions:

3) \(U\) \(M\)-subharmonic somewhere, \(U \in Msh\{0\}\)

4) For each fixed \(z \in D\) the function \(U(\lambda z)\) is harmonic in the circle \(\{\lambda \in C^n : \lambda z \in D\}\)

Then, the function \(U\) is pluriharmonic in \(D\).

The proof of the Theorem I. From the conditions of Theorem and the formula of Puasson, for each fixed number \(|\lambda| \leq 1\) and the point \(z \in D\) we know

\[
U(\lambda z) = \frac{1}{2\pi} \int_0^{2\pi} U(\tau z) \text{Re} \frac{\xi + \lambda}{\xi - \lambda} dt \quad \text{where} \quad \tau = e^{it}.
\]

As in the theorem I, firstly, we look at the case when \(U \in C^2(G)\). From the conditions of the theorem for each fixed \(\lambda \quad |\lambda| \leq 1\), in the neighborhood \(G\), having derived both sides of the equation (I.2) on \(Z\), we have

\[
\Delta \xi U(\lambda z) = \frac{1}{2\pi} \int_0^{2\pi} \Delta \xi U(\xi, z) \text{Re} \frac{\xi + \lambda}{\xi - \lambda} dt,
\]

and when

\[
\frac{1}{2\pi} \int_0^{2\pi} \Delta \xi U(\xi, z) \text{Re} \frac{\xi + \lambda}{\xi - \lambda} dt = \Delta \xi U(0) = 0
\]

\[
\psi(\lambda z) = \frac{1}{2\pi} \int_0^{2\pi} \Delta \xi U(\xi z) \text{Re} \frac{\xi + \lambda}{\xi - \lambda} dt
\]

Let’s suppose the function \(\psi(\lambda z)\) is holomorphic according to \(\lambda\) in the circle \(|\lambda| \leq 1\) and \(\psi(0, z) = 0\) according to (I.3).

Therefore, \(\psi(\lambda z)\) equals 0 in fixed \(z \in G\) or identically, or takes a neighborhood of zero in some neighborhood of zero. The latter cannot be done because \(\text{Re} \psi(\lambda z) \geq 0\).

This implies that \(\psi(\lambda z) = 0\) for any fixed \(\lambda, \quad |\lambda| \leq 1\) and \(z \in G\). So \(\Delta \xi U(\lambda z) = 0\) and, that is to say \(U\) began coordinates harmonically in the neighborhood.

Consequently \(U \in C^\infty \{0\}\) as a function and \(U\) will be pluriharmonic in \(D\) according to the formed theorem of Forelli.
According to the formula of Poisson and theorems of Fubini, we have
\[
(U(\lambda z, \Delta \phi)) = \int_G \frac{1}{2\pi} \int_0^{2\pi} \Delta_z U(\tau z) \Re \frac{\xi + \lambda}{\xi - \lambda} dt dV = \int_G U(\xi, z) \Delta_z \phi dV \Re \frac{\xi + \lambda}{\xi - \lambda} dt \geq 0
\]
\[\tau = e^{it}\], where
\[
\psi(\lambda z) = \frac{1}{2\pi} \int_0^{2\pi} \int_G U(\xi, z) \Delta_z \phi dV \frac{\xi + \lambda}{\xi - \lambda} dt
\]

It is clear that the function

Holomorphic to $\lambda$, the function $\psi(\lambda z)$ is either identically equal to 0 or transfers the neighborhood of zero to some neighborhood of zero. The latter is impossible because of $\Re \psi(\lambda z) \geq 0$ which is in $G$. From it $\psi(\lambda z) \equiv 0$ for each $|\lambda| < 1$ and $z \in G$.

Consequently $(U(l), \Delta_z \phi) = 0$ for any non-negative in G function $\phi$ of the category $C^\infty$. This means that $U(z)$ is harmonic in $G$, and, partially $U$, infinitely smooth in the neighborhood of zero. This implies that $U$ is pluraharmonic in D.

The theorem is proven.

The theorem 1 can be used to continue holomorphic function in the following pattern:

Corollary 1. Let the function $f$ given in the entire circular domain $D \subset C^n$ satisfies the following conditions:

1) $\Re f$ M – subharmonic in the neighborhood $G \ni 0$.
2) Stiction $f / l$ is holomorphic in $D \cap l$ for each complex line $l \ni 0$.

Then, $f$ is holomorphic according to the set of variables in $D$.

In fact, according to Theorem 1. $\Re f$ is pluraharmonic in $D$.

We take the full sphere $B = B(o, r) \subset D$ and in it the function $F \in O(B)$ is found which is $\Re F = \Re f$. Then the difference $\phi(z) = F(z) - f(z)$ has the following features: $\Re \phi \equiv 0$ and stiction $\phi / l$ is holomorphic in the neighborhood $l \cap B$. Consequently, $\phi / l \equiv C(l)$ is a constant depending perhaps on $l$. But $C(l) = \phi(0) = F(0) - f(0) = const$ from which $f = F - const$ and
is a holomorphic function in $B$. Using the theorem of Forelli, we obtain holomorphicity $f$
in the neighborhood $D$. Corollary is proven.

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