THE EXISTENCE OF SOLUTIONS FOR A CLASS OF IMPULSIVE FRACTIONAL Q-DIFFERENCE EQUATIONS

Shuyuan Wan, Yuqi Tang, Qi GE*
Department of Mathematics, Yanbian University, Yanji 133002, Jilin, CHINA
Correspondence should be addressed to Qi GE, geqi9688@163.com

ABSTRACT

In this paper, we prove the existence and uniqueness of solutions for a class of initial value problem for impulsive fractional \( q \)-difference equation of order \( 1 < \alpha \leq 2 \) by applying some well-known fixed point theorems. Some examples are presented to illustrate the main results.

MSC: 26A33; 39A13; 34A37

Keywords: \( q \)-calculus; impulsive fractional \( q \)-difference equations; existence; uniqueness.

INTRODUCTION

In recent years, the topic of \( q \)-calculus has attracted the attention of several researchers and a variety of new results on \( q \)-difference and fractional \( q \)-difference equations can be found in the papers [1-13] and the references cited therein. In [14] the notions of \( q_k \)-derivative and \( q_k \)-integral of a function \( f: J_k := [t_k, t_{k+1}] \to \mathbb{R} \) have been introduced and their basic properties was proved. As applications existence and uniqueness results for initial value problems for first and second order impulsive \( q_k \)-difference equations are proved. In [15], the authors applied the concepts of quantum calculus developed in [14] to study a class of boundary value problem of ordinary impulsive \( q_k \)-integro-difference equations, some existence and uniqueness results for this problem were proved by using a variety of fixed point theorems. In [16] the authors used the \( q \)-shifting operator to develop the new concepts of fractional quantum calculus such as the Riemann–Liouville fractional derivative and integral and their properties. They also formulated the existence and uniqueness results for some classes of first and second orders impulsive fractional \( q \)-difference equations. Inspired by [16], in this paper, we study the existence and uniqueness of solutions for the following initial value problem for impulsive fractional \( q \)-difference equation of order \( 1 < \alpha \leq 2 \) the form

\[
\begin{cases}
\iota_1 D^\alpha_{t_0} x(t) = f(t, x(t)), & t \in J, t \neq t_k \\
\Delta x(t_k) = \phi_k(x(t_k)), & k = 1, 2, \ldots, m, \\
\Delta^\gamma x(t_k) = \phi^*_{k}(x(t_k)), & k = 1, 2, \ldots, m, \\
x(0) = 0, & \iota_0^\alpha D^\alpha_{t_0} x(0) = \beta_0 D^\alpha_{t_0} x(\eta),
\end{cases}
\]

(1.1)

where \( J = [0, T] \), \( 0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots < t_m < t_{m+1} = T \), \( J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}] \), \( k = 1, 2, \ldots, m \), \( \iota_1 D^\alpha_{t_0} \) and \( \iota_0^\alpha D^\alpha_{t_0} \) respectively are the Riemann–Liouville fractional \( q \)-difference of order \( \alpha \) and \( \alpha - 1 \) on interval \( J_0 \), \( 0 < q_k < 1 \) for \( k = 1, 2, \ldots, m \), \( f: J \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( \phi_k, \phi^*_k \in C(R, R) \) for \( k = 1, 2, \ldots, m \). The notation \( \Delta x(t_k) \) and \( \Delta^\gamma x(t_k) \) are defined by

\[
\begin{align*}
\Delta x(t_k) &= \iota_1^\alpha L^\alpha_{t_k} x(t_k^+) - \iota_1^\alpha L^\alpha_{t_k} x(t_k^-), & k = 1, 2, \ldots, m, \\
\Delta^\gamma x(t_k) &= \iota_1^\alpha L^\alpha_{t_k} x(t_k^+) - \iota_1^\alpha L^\alpha_{t_k} x(t_k^-), & k = 1, 2, \ldots, m,
\end{align*}
\]

(1.2)

where \( \iota_1^\alpha L^\alpha_{t_k} \) and \( \iota_1^\alpha L^\alpha_{t_k} \) respectively are the Riemann-Liouville fractional \( q \)-integral of order \( 1 - \alpha \) and \( 2 - \alpha \) on \( J_k \). \( \beta \in \mathbb{R}, k_0 \in \{1, 2, \ldots, m\}, \eta \in (t_{k_0}, t_{k_0+1}] \).
Preliminaries

This section is devoted to some basic concepts such as $q$-shifting operator, Riemann–Liouville fractional $q$-integral and $q$-difference on a given interval. The presentation here can be found in, for example, [16,17].

We define a $q$-shifting operator as

$$\Phi_q(m) = q^m + (1-q)a.$$  

The power of $q$-shifting operator is defined as

$$\Phi_q(m)^{(n-m)} = 1, \Phi_q(m)^{(k)} = \prod_{i=0}^{k-1} (n - \Phi_q(m)), k \in \mathbb{N} \cup \{\infty\},$$ 

More generally, if $\gamma \in \mathbb{R}$, then

$$\Phi_q(m)^{\gamma} = \prod_{i=0}^{\gamma} 1 - \frac{1}{1 - q} \Phi_q(m/n).$$

**Definition 2.1.** The fractional $q$-derivative of Riemann–Liouville type of order $\nu \geq 0$ on interval $[a,b]$ is defined by $(\_\nu^D_q t f)(t) = f(t)$ and

$$(\_\nu I_q f)(t) = (\_\nu^D_q t^\nu f)(t), \quad v > 0,$$

where $l$ is the smallest integer greater than or equal to $v$.

**Definition 2.2.** Let $\alpha \geq 0$ and $f$ be a function defined on $[a,b]$. The fractional $q$-integral of Riemann–Liouville type is given by $(\_\alpha^I_q t f)(t) = f(t)$ and

$$(\_\alpha I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - \Phi_q(s))^{\alpha-1} f(s) \, ds, \quad s > 0, t \in [a,b].$$

From [16], we have the following formulas for $t \in [a,b], \alpha > 0, \beta \in \mathbb{R}$:

$$\Phi_q(t - a)^{\beta} = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta - \alpha + 1)} (t - a)^{\beta - \alpha}, \quad \Phi_q(t - a)^{\alpha} = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)} (t - a)^{\beta + \alpha}.$$  

**Lemma 2.3.** Let $\alpha, \beta \in \mathbb{R}^{+}$ and $f$ be a continuous function on $[a,b], a \geq 0$. The Riemann–Liouville fractional $q$-integral has the following semi-group property

$$\_\alpha^I_q \_\beta^I_q f(t) = \_\alpha^I_q \_\beta^I_q f(t) = \_\alpha^I_q f(t).$$

**Lemma 2.4.** Let $f$ be a $q$-integrable function on $[a,b]$. Then the following equality holds

$$\_\alpha^D_q \_\beta^I_q f(t) = f(t).$$

**Lemma 2.5.** Let $t \alpha > 0$ and $p$ be a positive integer. Then for $t \in [a,b]$ the following equality holds

$$\_\alpha^I_q \_\beta^D_q f(t) = \_\alpha^D_q \_\beta^I_q f(t) = \sum_{k=0}^{\infty} \frac{(t - a)^{\alpha - p + k}}{\Gamma_q(\alpha + k - p + 1)} D_q^k f(a).$$

**Lemma 2.6.** ([18]) Let $E$ be a Banach space. Assume that $\Omega$ is an open bounded subset of $E$ with $\theta \in \Omega$ and let $T : \Omega \rightarrow E$ be a completely continuous operator such that

$$\|Tu\| \leq \|u\|, \forall u \in \partial \Omega.$$  

Then $T$ has a fixed point in $\Omega$.

**Lemma 2.7.** ([18]). Let $E$ be a Banach space. Assume that $T : E \rightarrow E$ is a completely continuous operator and the set $V = \{u \in E | \mu |u|, 0 < \mu < 1\}$ is bounded. Then $T$ has a fixed point in $E$.

Let $PC(J, R) = \{x : J \rightarrow R : x(t) \text{ is continuous everywhere except for some } t_i \text{ at which } x(t_i) \text{ and } x(t_i^+)\}$
exist and \( x(t_k) = x(t_k) \), \( k = 1, 2, 3, \ldots, m \). For \( \gamma \in \mathbb{R}^+ \), we introduce the space \( C_{\gamma,k}(J_k, \mathbb{R}) = \{ x : J_k \to \mathbb{R} : (t-t_j)^\gamma x(t) \in C(J_k, \mathbb{R}) \} \) with the norm \( \| x \|_{C_{\gamma,k}} = \sup_{t \in J_k} \| (t-t_j)^\gamma x(t) \| \) and \( PC_{\gamma,k}(J_k, \mathbb{R}) = \{ x : J_k \to \mathbb{R} : \) for each \( j \in J_k \) and \( (t-t_j)^\gamma x(t) \in C(J_k, \mathbb{R}), k = 0, 1, 2, \ldots, m \) with the norm \( \| x \|_{PC_{\gamma,k}} = \max\{ \sup_{t \in J_k} \| (t-t_j)^\gamma x(t) \| : k = 0, 1, 2, \ldots, m \} \).

Clearly \( PC_{\gamma}(J_k, \mathbb{R}) \) is a Banach space.

**Lemma 2.8.** If \( x \in PC_{\gamma}(J_k, \mathbb{R}) \) is a solution of (1.1), then for any \( t \in J_k, k = 1, 2, \ldots, m \),

\[
x(t) = m_0(t-t_k)^{\alpha-2} + \frac{m_1(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} + \frac{m_2(t-t_k)^{\alpha}}{\Gamma(\alpha)} f(t, x(t)),
\]

(2.1)

Where

\[
m_0 = \frac{\beta_1}{1-\beta} \left[ \int_{t_k-\eta}^{t_k} t^{\alpha-1} f(s, x(s))(\eta) + \sum_{0<\tau<\eta} \left( \int_{t_k}^{\tau} t^{\alpha-1} f(s, x(s))(\tau) + \varphi_j(x(t_j)) \right) \right]
\]

\[+ \sum_{0<\tau<\eta} \sum_{0<\tau<\eta} \left( \int_{t_k}^{\tau} t^{\alpha-1} f(s, x(s))(\tau) + \varphi_j(x(t_j)) \right) + \sum_{0<\tau<\eta} \left( \int_{t_k}^{\tau} t^{\alpha-1} f(s, x(s))(\tau) + \varphi_j(x(t_j)) \right) \]

(2.2)

\[
m_2 = \frac{\beta_1}{1-\beta} \left[ \int_{t_k-\eta}^{t_k} t^{\alpha-1} f(s, x(s))(\eta) + \sum_{0<\tau<\eta} \left( \int_{t_k}^{\tau} t^{\alpha-1} f(s, x(s))(\tau) + \varphi_j(x(t_j)) \right) \right]
\]

\[+ \sum_{0<\tau<\eta} \left( \int_{t_k}^{\tau} t^{\alpha-1} f(s, x(s))(\tau) + \varphi_j(x(t_j)) \right) \]

(2.3)

With \( \sum_{0<\eta} \eta = 0 \).

**Proof.** For \( t \in J_m \), taking the Riemann-Liouville fractional \( q_0 \)-integral of order \( \alpha \) for the first equation of (1.1) and using Definition 2.1 with Lemma 2.5, we get

\[
x(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha)} C_0 + \int_{t_m}^{t} \frac{t^{\alpha-1}}{\Gamma(\alpha)} C_1 + \frac{t^{\alpha}}{\Gamma(\alpha)} \int_{t_m}^{t} f(t, x(t)) \]

(2.4)

where \( C_0 = 0 t^{\alpha-\alpha} x(0) \) and \( C_1 = 0 t^{\alpha-\alpha} x(0) \). The first initial condition of (1.1) implies that \( C_0 = 0 \). Taking the Riemann-Liouville fractional \( q_0 \)-derivative of order \( \alpha - 1 \) for (2.4) on \( J_m \), we have

\[
0 t^{\alpha-1} D_{q_0}^{\alpha-1} x(t) = C_1 + 0 t^{\alpha} f(t, x(t)),
\]

And \( 0 t^{\alpha-1} x(0) = C_1 \). Therefore, (2.4) can be written as

\[
x(t) = \frac{t^{\alpha}}{\Gamma(\alpha)} C_1 + 0 t^{\alpha} f(t, x(t)).
\]

(2.5)

Applying the Riemann-Liouville fractional \( q_0 \)-derivative of orders \( 1-\alpha \) and \( 2-\alpha \) for (2.5) at \( t = t_k \), we have

\[
0 t^{\alpha-\alpha} x(t_k) = C_1 + 0 t^{\alpha-\alpha} f(s, x(s))(t_k), \quad 0 t^{2-\alpha} x(t_k) = C_1 + 0 t^{2-\alpha} f(s, x(s))(t_k).
\]

(2.6)

For \( t \in J_k, k = 1,2, \ldots, m \), Riemann-Liouville fractional \( q_0 \)-integrating (1.1), we obtain

\[
x(t) = \frac{(t-t_k)^{\alpha-2}}{\Gamma(\alpha)} C_1 + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} f(t, x(t)),
\]

(2.7)

Using the jump conditions of equation (1.1) with (2.6)-(2.7) for \( t \in J_k \), we get

\[
x(t) = \frac{(t-t_k)^{\alpha-2}}{\Gamma(\alpha)} [C_1 + 0 t^{\alpha} f(s, x(s))(t_k) + \varphi_j(x(t_k))] + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} [C_1 + 0 t^{\alpha} f(s, x(s))(t_k) + \varphi_j(x(t_k))] + 0 t^{\alpha} f(t, x(t)).
\]

Repeating the above process, for \( t \in J_k = (t_k, t_{k+1}) \), we obtain
\[ x(t) = \frac{(t-t_0)^{\alpha-2}}{\Gamma_{q_0}(\alpha-1)} \left[ C_i + \sum_{0<t_i<\tau} \sum_{t_i<\tau} (t_{i+1} - t_{i-1}) \left( \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right) \right] + \sum_{0<t_i<\tau} \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right] + \frac{(t-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} \left[ C_i \right. \]
\[ \left. + \sum_{0<t_i<\tau} \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right] + \int_{t_i}^{t_{i+1}} f(t,x(t)) \right]. \]

Taking the Riemann-Liouville fractional \( q_0 \)-derivative of order \( \alpha-1 \) for (2.8) and using \( \Gamma_{q_0}(0) = \infty \),
it follows that
\[ \int_{t_i}^{t_{i+1}} \frac{(t-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} \left[ C_i + \sum_{0<t_i<\tau} \sum_{t_i<\tau} (t_{i+1} - t_{i-1}) \left( \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right) \right] + \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right] + \int_{t_i}^{t_{i+1}} f(t,x(t)) \right]. \]

For \( k_0 \in \{1,2,\ldots,m\} \), \( \eta \in (t_{k_0},t_{k_0+1}] \), we have
\[ \int_{t_{k_0}}^{t_{k_0+1}} \frac{(t-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} \left[ C_i + \sum_{0<t_i<\tau} \sum_{t_i<\tau} (t_{i+1} - t_{i-1}) \left( \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right) \right] + \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right] + \int_{t_i}^{t_{i+1}} f(s,x(s))(\eta) \right]. \]

The initial condition \( \frac{D^{\alpha-1}x(0)}{q_0}=\beta_k \int_{t_{k_0}}^{t_{k_0+1}} \frac{(t-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} \left[ C_i + \sum_{0<t_i<\tau} \sum_{t_i<\tau} (t_{i+1} - t_{i-1}) \left( \int_{t_i}^{t_{i+1}} f(s,x(s))(t_i) + \varphi_j(x(t_i)) \right) \right] + \int_{t_i}^{t_{i+1}} f(s,x(s))(\eta) \right] \]

Substituting the value of \( C_i \) in (2.8), we obtain (2.1). Conversely, assume that \( x \) is a solution of the impulsive fractional integral equation (2.1), then by a direct computation, it follows that the solution given by (2.1) satisfies equation (1.1). This completes the proof.

**Main results**

This section deals with the existence and uniqueness of solutions for the equation (1.1). In view of Lemma 2.8, we define an operator \( A: PC(J,R) \to PC(J,R) \) by
\[ (Ax)(t) = \frac{m_1(t-t_0)^{\alpha-2}}{\Gamma_{q_0}(\alpha-1)} + \frac{m_2(t-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} + \frac{A}{\Gamma_{q_0}(\alpha)} f(t,x(t)) \]
where \( m_1, m_2 \) are given by (2.2) and (2.3).

**Theorem 3.1.** Let \( \lim_{t \to 0^+} \frac{f(t,x)}{x} = 0, \lim_{t \to 0^+} \frac{\partial f(x)}{x} = 0 \) and \( \lim_{t \to 0^+} \frac{\partial f(x)}{x} = 0 \) \((k=1,2,\ldots,m)\), then equation (1.1) has at least one solution.

Proof. To show that \( Ax \in PC(J,R) \) for \( x \in PC(J,R) \), we suppose \( t_1, t_2 \in J_k \), and \( t_1 > t_2 \), then
\[ \left| (t_1 - t_0)^{\alpha-1} A(x(t_1)) - (t_2 - t_0)^{\alpha-1} A(x(t_2)) \right| \]
\[ = \left| (t_1 - t_0)^{\alpha-1} \frac{m_1(t_1-t_0)^{\alpha-2}}{\Gamma_{q_0}(\alpha-1)} + \frac{m_2(t_1-t_0)^{\alpha-1}}{\Gamma_{q_0}(\alpha)} + \frac{A}{\Gamma_{q_0}(\alpha)} f(t,x(s))(t_1) \right] \]
\[-(z_t - t_i)^{\alpha-2}\left[\frac{m_k(z_t - t_i)^{\alpha-2}}{\Gamma_{\alpha}(\alpha-1)} + \sum_{i=1}^{k}(t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\right]\]

\[\leq \left(\frac{(z_t - t_i)^{\alpha-2} - (z_t - t_i)^{\alpha-2}}{\Gamma_{\alpha}(\alpha-1)}\right)\left[\frac{\beta T_{\alpha}}{1 - \beta}T_{\alpha}^a f(x, x(s)|z_t)\right] + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

\[\leq \left(\frac{(z_t - t_i)^{\alpha-2} - (z_t - t_i)^{\alpha-2}}{\Gamma_{\alpha}(\alpha-1)}\right)\left[\frac{\beta T_{\alpha}}{1 - \beta}T_{\alpha}^a f(x, x(s)|z_t)\right] + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

As \(z_t \to z_{t_i}\), we have \(|z_t - z_{t_i}| = (z_t - z_{t_i})r_{\alpha} \to 0\) for each \(k = 0, 1, 2, \ldots, m\). Therefore, we get \(Ax \in PC_{\alpha}(J, R)\). Now we show that the operator \(A: PC_{\alpha}(J, R) \to PC_{\alpha}(J, R)\) is completely continuous, not that \(A\) is continuous in view of continuity of \(f, \varphi\) and \(\varphi^\ast\). Let \(B \in PC_{\alpha}(J, R)\) be bounded. Then, there exist positive constants \(L_{\alpha} > 0 \ (\alpha = 1, 2, 3)\) such that \(|f(t, x)| \leq L_{\alpha}, |\varphi(t, x)| \leq L_{\alpha}, |\varphi^\ast(t, x)| \leq L_{\alpha}, \forall x \in B\). Thus, \(\forall x \in B\),

We have

\[|m_t| \leq \frac{\beta T_{\alpha}}{1 - \beta}T_{\alpha}^a f(x, x(s)|z_t) + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

\[\frac{1}{\Gamma_{\alpha}(\alpha)}\int_{t_i}^{t_{i+1}}[\frac{(z_t - t_i)^{\alpha-2} - (z_t - t_i)^{\alpha-2}}{\Gamma_{\alpha}(\alpha-1)} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)]dt\]

\[\leq \frac{\beta T_{\alpha}}{1 - \beta}L_{\alpha} + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

\[\mid m_t \mid \leq \frac{\beta T_{\alpha}}{1 - \beta}L_{\alpha} + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

Therefore,

\[(t - t_i)^{\alpha-2} \left[\frac{(t - t_i)^{\alpha-2}}{\Gamma_{\alpha}(\alpha-1)} + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\right]\]

\[\leq \frac{\beta T_{\alpha}}{1 - \beta}L_{\alpha} + \sum_{i=1}^{k} \sum_{j=1}^{k} (t_i, t_{i+1})^{\alpha-1} + \frac{1}{\Gamma_{\alpha}(\alpha)}T_{\alpha}^a f(x, x(s)|z_t)\]

which implies that

\[\text{Vol. 4 No. 1, 2017}
\]
\[
\|Ax(t)\| \leq \frac{T^{\alpha-2}}{\Gamma_\alpha(\alpha-1)} \left[ \frac{\beta T}{1-\beta} \left( L_2 \eta + mL_2 + 2L_mT^2 + \frac{m^2T\eta}{2} \right) + \frac{T^{\alpha-1}}{\Gamma_\alpha(\alpha)} \right] \left[ \frac{\delta \eta + m\delta_2 + m\delta_3 + 2\delta_mT^2 + \frac{m^2T\delta_2}{2}}{1-\beta} \right] + \frac{T^{\alpha-1}}{\Gamma_\alpha(\alpha)} \left[ \frac{\beta T}{1-\beta} \left( L_2 \eta + mL_2 + L_T + mL_2 + \frac{L_T^2T^2}{\Gamma_\alpha(\alpha+1)} \right) \right] \leq L.
\]

On the other hand, for any \(t, t_2 \in J_k\), with \(t_i < t_2, 0 \leq k \leq m\), we have
\[
\left| (t_2 - t_1)^\gamma (Ax)(t) - (t_1 - t_1)^\gamma (Ax)(t_1) \right|
\leq \frac{T^{\alpha-2}}{\Gamma_\alpha(\alpha-1)} \left[ \frac{\beta T}{1-\beta} \left( L_2 \eta + k_2L_2 + kL_2 + 2L_2kT^2 + \frac{k^2T^2L_2}{2} \right) + \frac{T^{\alpha-1}}{\Gamma_\alpha(\alpha)} \right] \left[ \frac{\delta \eta + k_2L_2 + L_T + kL_2 + k^2T^2L_2}{1-\beta} \right]
+ \left| (t_2 - t_1)^\gamma \alpha I_a^\alpha (s, x(s))(t_2) - (t_1 - t_1)^\gamma \alpha I_a^\alpha (s, x(s))(t_1) \right| \rightarrow 0 \quad (t_i \rightarrow t_2).
\]

This implies that \(A\) is equscontinuous on all the subintervals \(J_k, k = 0, 1, \ldots, m\). Thus, by Arzela–Ascoli Theorem, it follows that \(A : PC(J, R) \rightarrow PC(J, R)\) is completely continuous.

Now, in view of \(\lim_{x \rightarrow 0} f(t, x) = 0, \lim_{x \rightarrow 0} \phi_1(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \phi_2(x) = 0 \quad (k = 1, 2, \ldots, m)\), there exists a constant \(r > 0\) such that \(|f(t, x)| \leq \delta_1|x|, \quad |\phi_1(x)| \leq \delta_2|x|, \quad |\phi_2(x)| \leq \delta_3|x|, \quad \text{for} \quad 0 < |x| < r, \quad \text{where} \quad \delta_i > 0 \quad (i = 1, 2, 3)\) satisfy
\[
\frac{T^{\alpha-2}}{\Gamma_\alpha(\alpha-1)} \left[ \frac{\beta T}{1-\beta} \left( \delta \eta + m\delta_2 + m\delta_3 + 2\delta_2 T^2 + \frac{m^2T\delta_2}{2} \right) + \frac{T^{\alpha-1}}{\Gamma_\alpha(\alpha)} \left[ \frac{\beta T}{1-\beta} \left( \delta \eta + m\delta_2 + \delta T + m\delta_1 \right) + \frac{\delta T^a}{\Gamma_\alpha(\alpha+1)} \right] \right] \leq 1.
\]
Define \(\Omega = \{x \in PC(J, R) : \|x\| < r\}\) and take \(x \in PC(J, R)\) such that \(\|x\| = r\) so that \(x \in \partial \Omega\). Then, by the process used to obtain (3.1), we have
\[
(t - t_1)^\gamma \|Ax(t)\| \leq \frac{T^{\alpha-2}}{\Gamma_\alpha(\alpha-1)} \left[ \frac{\beta T}{1-\beta} \left( \delta \eta + m\delta_2 + m\delta_3 + 2\delta_2 T^2 + \frac{m^2T\delta_2}{2} \right) + \frac{T^{\alpha-1}}{\Gamma_\alpha(\alpha)} \left[ \frac{\beta T}{1-\beta} \left( \delta \eta + m\delta_2 + \delta T + m\delta_1 \right) + \frac{\delta T^a}{\Gamma_\alpha(\alpha+1)} \right] \right] \|x\| \leq \|x\|,
\]
which implies that \(\|Ax(t)\| \leq \|x\|, x \in \partial \Omega\).

Therefore, by Lemma 2.6, the operator \(A\) has at least one fixed point, which in turn implies that (1.1) has at least one solution \(x \in \partial \Omega\). This completes the proof.

**Theorem 3.2.** Assume that
\[(H_1) \text{ there exist positive constants } L_i \ (i = 1, 2, 3) \text{ such that} \quad |f(t, x)| \leq L_1, \quad |\phi_1(x)| \leq L_2, \quad |\phi_2(x)| \leq L_3 \text{ for } t \in J, x \in R \quad \text{and} \quad k = 1, 2, \ldots, m.\]

Then equation (1.1) has at least one solution.

**Proof.** As shown in Theorem 3.1, the operator \(A : PC(J, R) \rightarrow PC(J, R)\) is completely continuous. Now, we show the set \(V = \{x \in PC(J, R) : x = \mu Ax, 0 < \mu < 1\}\) is bounded.

Let \(x \in V\), then \(x = \mu Ax, 0 < \mu < 1\). For any \(t \in J\), we have
\[
x(t) = \frac{m_{\alpha} (t - t_1)^{\alpha-2}}{\Gamma_\alpha(\alpha-1)} + \frac{m_{\alpha} (t - t_1)^{\alpha-1}}{\Gamma_\alpha(\alpha)} + \mu \alpha I_a^\alpha (s, x(t)) \quad \text{for} \quad x(t) = f(t, x(t)),
\]
where \(m_i, m_i\) are given by (2.2) and (2.3). Combining \((H_1)\) and (3.2), we obtain
\((t-t_0)^y |x(t)| \leq \frac{\mu|n_1|(t-t_0)^{\gamma-1}}{\Gamma_{q_1}(\alpha-1)} + \frac{\mu|n_2|(t-t_0)^{\gamma-1}}{\Gamma_{q_2}(\alpha)} + \mu(t-t_0)^{\eta_1} f(x(t)) \]

\[ \leq \frac{\mu T^{\gamma-1}}{\Gamma_{q_0}(\alpha-1)} \left[ \begin{array}{c} \frac{\beta T \eta + mL_2}{1-\beta} \right] t \right] + \frac{\mu T^{\gamma-1}}{\Gamma_{q_0}(\alpha-1)} \left[ \begin{array}{c} \frac{\beta T \eta + mL_2}{1-\beta} \right] t \right] \]

Thus, for any \( t \in J \), it follows that \( |x| \leq L \). So, the set \( V \) is bounded. Therefore, by the conclusion of Lemma 2.7, the operator \( A \) has at least one fixed point. This implies that (1.1) has at least one solution. This completes the proof.

**Theorem 3.3. Assume that**

(H2) there exist positive constants \( N_i (i = 1, 2, 3) \) such that

\[ |f(t,x) - f(t,y)| \leq N_1 |x - y|, \quad \eta \leq N_2 |x - y|, \quad \eta \leq N_3 |x - y| \]

for \( t \in J, x, y \in \mathbb{R} \) and \( k = 1, 2, \ldots, m \).

Then equation (1.1) has a unique solution if

\[ A = T^- \left( \begin{array}{c} \frac{\beta}{1-\beta} \left[ \begin{array}{c} \eta \| f(s,x(s)) - f(s,y(s)) \| \right] \\
+ \sum_{j=1}^{k} I^1_{q_j}, \right] \right) \]

\[ + \frac{\mu T^{\gamma-1}}{\Gamma_{q_0}(\alpha-1)} \left[ \begin{array}{c} \frac{\beta T \eta + mL_2}{1-\beta} \right] t \right] \]

\[ + \frac{\mu T^{\gamma-1}}{\Gamma_{q_0}(\alpha-1)} \left[ \begin{array}{c} \frac{\beta T \eta + mL_2}{1-\beta} \right] t \right] \]

**Proof.** For \( x, y \in PC(J, \mathbb{R}) \), we have

\[ (t-t_0)^y |Ax(t) - Ay(t)| \leq \frac{(t-t_0)^{\gamma-1}}{\Gamma_{q_0}(\alpha-1)} \left[ \begin{array}{c} \frac{\beta T \eta + mL_2}{1-\beta} \right] t \right] \]

\[ + \sum_{j=1}^{k} I^1_{q_j}, \]
\[
\begin{align*}
\leq \frac{T^*}{T} \left\| \frac{\beta}{1-\beta} (N_{ij} + k_i N_j) (T + 1) + k N_j + N_i (1 + T + 2 k T^2) + N_j (k + \frac{m^2 T^2}{2}) \right\| x - y \|_{\text{pc}}, \\
\leq \frac{T^*}{T} \left\| \frac{\beta}{1-\beta} (N_{ij} + m N_j) (T + 1) + m N_j + N_i (1 + T + 2 m T^2) + N_j (m + \frac{m^2 T^2}{2}) \right\| x - y \|_{\text{pc}}, \\
< A \| x - y \|_{\text{pc}},
\end{align*}
\]

where \(A\) is given by (3.3). Thus, \(\| Ax - Ay \|_{\text{pc}} \leq A \| x - y \|_{\text{pc}}\). As \(A < 1\), therefore, \(A\) is a contraction. Hence, by the contraction mapping principle, equation (1.1) has a unique solution.

**Examples**

**Example 4.1.** Consider the following impulsive fractional \(q\)-difference initial value problem:

\[
\begin{align*}
\Delta x(t_k) &= k - k \cos x(t_k), \quad k = 1, 2, \ldots, 10, \quad t = [0, \frac{11}{10}], t \neq t_k, \\
\Delta' x(t_k) &= k \sin^3 x(t_k), \quad k = 1, 2, \ldots, 10, \quad t = \frac{k}{10}, \\
x(0) &= 0, \quad x'\left(\frac{1}{2}\right) = \frac{2}{3} \frac{D^\frac{1}{2}}{\pi^2} x(1)
\end{align*}
\]

Here \(\alpha = 3/2, \quad q_k = (k^3 - 3k + 7)/(2k^4 + k + 8), \quad k = 1, 2, \ldots, 10, \quad m = 10, \quad T = 11/10, \quad \beta = 2/3, \quad k_0 = 2, \quad \eta = 1/4, \quad f(t, x(t)) = t^2 \arctan^2 x(t) + e^{x^3(t)} \cdot \phi_2(x(t)) = k - k \cos x(t_k), \quad \phi_2(x(t_k)) = k \sin^3 x(t_k)\),

Clearly, all the assumptions of Theorem 3.1 are satisfied. Thus, by the conclusion of Theorem 3.1, the impulsive fractional \(q\)-difference initial value problem 4.1 has at least one solution.

**Example 4.2.** Consider the following impulsive fractional \(q\)-difference initial value problem:

\[
\begin{align*}
\Delta x(t_k) &= k + 3 k^2 \cos^2 x(t_k), \quad k = 1, 2, \ldots, 9, \quad t = \frac{k}{10}, \\
\Delta' x(t_k) &= k \sin(4 + e^{10t}), \quad k = 1, 2, \ldots, 9, \quad t = \frac{k}{10}, \\
x(0) &= 0, \quad x'\left(\frac{1}{2}\right) = \frac{2}{3} \frac{D^\frac{1}{2}}{\pi^2} x(1)
\end{align*}
\]

Here \(\alpha = 3/2, \quad q_k = (k^3 - 3k + 7)/(2k^4 + k + 8), \quad k = 1, 2, \ldots, 9, \quad m = 9, \quad T = 1, \quad \beta = 2/3, \quad k_0 = 2, \quad \eta = 1/4, \quad f(t, x(t)) = e^{x^3(t)} \cdot \phi_2(x(t)) = k + 3 k^2 \cos^2 x(t_k), \quad \phi_2(x(t_k)) = k \sin(4 + e^{10t})\),

Clearly \(I = c, \quad L = 36, \quad L = 9\) and the conditions of Theorem 3.2 can readily be verified. Therefore, the conclusion of Theorem 3.2 applies to the impulsive fractional \(q\)-difference initial value problem 4.2.

**REFERENCES**