COMMON FIXED POINT THEOREM IN $b_2$-METRIC SPACES

Jinxing Cui$^1$ & Linan Zhong$^1$

(Department of Mathematics, Yanbian University, Yanji, 133002, P.R. China)

*Corresponding author: Linan Zhong. Email: zhonglinan2000@126.com.

ABSTRACT

We establish a unique common fixed point theorem for two pair of weekly compatible maps satisfying a contractive condition in a complete $b_2$-metric space. When the following have been proved, I recommend it to be published, which extends and generalizes some known results in metric space to $b_2$-metric space.

Keywords: Common fixed point; complete $b_2$-metric space; weekly compatible maps.

1 Introduction

Fixed point theory has been studied by many authors for its useful function in a variety of areas. In 1992, a polish mathematician, Banach, proved a theorem known as Banach contraction principle [1]. This principle presents useful results in nonlinear analysis, functional analysis and topology. The concept of weakly commuting has been introduced by Sesssa S [2]. Years later, Gerald Jungck [3] introduced weakly compatible mappings, which are more generalized commuting mappings.

In this paper, we present fixed point results for two pair of mappings satisfying a contractive type condition by using the concept of weakly compatible mappings in a complete generalized metric space, which is called $b_2$-metric space [5] and this space was generalized from both 2-metric space [6-8] and b-metric space [9-10].

2 Preliminaries

The following definitions will be needed to present before giving our results.

Definition 2.1 [2] Let $f$ and $g$ be two self-maps on a set $X$. Maps $f$ and $g$ are said to be commuting if $fgx=gfx$ for all $x \in X$.

Definition 2.2 [4] Let $f$ and $g$ be two self-maps on a set $X$. If $fx=gx$, for some $x$ of $X$, then $x$ is called coincidence point of $f$ and $g$.

Definition 2.3 [4] Let $f$ and $g$ be two self-maps defined on a set $X$. Then $f$ and $g$ are said to be weakly compatible if they commute at coincidence points. That is, if $fx=gx$ for some $x \in X$, then $fgx=gfx$.

Lemma 2.4 [4] Let $f$ and $g$ be weakly compatible self mappings of a set $X$. If $f$ and $g$ have a unique point of coincidence, that is, $\omega=fx=gx$, then $\omega$ is the unique common fixed point of $f$ and $g$.

Definition 2.5 [5] Let $X$ be a nonempty set, $s \geq 1$ be a real number and let $d : X \times X \times X \rightarrow R$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$. 

2. If at least two of three points \( x, y, z \) are the same, then \( d(x, y, z) = 0 \).

3. The symmetry:
\[
d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)
\]
for all \( x, y, z \in X \).

1. The rectangle inequality:
\[
d(x, y, z) \leq [d(x, y, a) + d(y, z, a) + d(z, x, a)], \quad \text{for all} \quad x, y, z, a \in X.
\]

Then \( d \) is called a \( b_2 \) metric on \( X \) and \((X,d)\) is called a \( b_2 \) metric space with parameter \( s \). Obviously, for \( s = 1 \), \( b_2 \) metric reduces to 2-metric.

**Definition 2.6** [5] Let \( \{x_n\} \) be a sequence in a \( b_2 \) metric space \((X,d)\).

1. A sequence \( \{x_n\} \) is said to be \( b_2 \)-convergent to \( x \in X \), written as \( \lim_{n \to \infty} x_n = x \), if all \( a \in X \) \( \lim_{n \to \infty} d(x_n, x, a) = 0 \).

2. \( \{x_n\} \) is Cauchy sequence if and only if \( d(x_n, x_m, a) \to 0 \), when \( n, m \to \infty \). for all \( a \in X \).

3. \((X, d)\) is said to be -complete if every \( b_2 \)-Cauchy sequence is a \( b_2 \)-convergent sequence.

**Definition 2.7** [5] Let \((X, d)\) and \((X', d')\) be two \( b_2 \)-metric spaces and let \( f : X \to X' \) be a mapping. Then \( f \) is said to be \( b_2 \)-continuous, at a point \( z \in X \) if for a given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( x \in X \) and \( d(z, x, a) < \delta \) for all \( a \in X \) imply that \( d'(fz, fx, a) < \varepsilon \). The mapping \( f \) is \( b_2 \)-continuous on \( X \) if it is \( b_2 \)-continuous at all \( z \in X \).

**Definition 2.8** [5] Let \((X, d)\) and \((X', d')\) be two \( b_2 \)-metric spaces. Then a mapping \( f : X \to X' \) is \( b_2 \)-continuous at a point \( x \in X' \) if and only if it is \( b_2 \)-sequentially continuous at \( x \); that is, whenever \( \{x_n\} \) is \( b_2 \)-convergent to \( x \), \( \{fx_n\} \) is \( b_2 \)-convergent to \( f(x) \).

**Definition 2.9** [6-8] Let \( X \) be a nonempty set and let \( d : X \times X \times X \to R \) be a map satisfying the following conditions:

1. For every pair of distinct points \( x, y \in X \), there exists a point \( z \in X \) such that \( d(x, y, z) \neq 0 \).

2. If at least two of three points \( x, y, z \) are the same, then \( d(x, y, z) = 0 \).

3. The symmetry:
\[
d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)
\]
for all \( x, y, z \in X \).

4. The rectangle inequality:
\[
d(x, y, z) \leq d(x, y, a) + d(y, z, a) + d(z, x, a) \quad \text{for all} \quad x, y, z, a \in X.
\]

Then \( d \) is called a 2 metric on \( X \) and \((X,d)\) is called a 2 metric space.

**Definition 2.10** [9-10] Let \( X \) be a nonempty set and \( s \geq 1 \) be a given real number. A function \( d : X \times X \to R^+ \) is a \( b \) metric on \( X \) if for all \( x, y, z \in X \), the following conditions hold:

1. \( d(x, y) = 0 \) if and only if \( x = y \).
3. \( d(x, y) = d(y, x) \).
4. \( d(x, y) \leq s[d(x, y) + d(y, z)] \).

In this case, the pair \((X, d)\) is called a \(b\) metric space.

3 Main results

**Theorem 3.1.** Let \((X, d)\) be a complete \(b_2\)-metric space, and \(P, Q, S, T : X \to X\) are four mappings, satisfying the following conditions:

(a) \( T(X) \subseteq P(X) \) and \( S(X) \subseteq Q(X) \); Both \( P \) and \( Q \) are surjections.

(b) \( d(Sx, Ty, a) \leq c(\lambda(x, y, a)) \).

Where \( \lambda(x, y, a) = \max\{d(Px, Qx, a), d(Px, Sx, a), d(Qx, Ty, a)\} \) for all \( x, y \in X \) and \( 0 \leq c < \frac{1}{s} \).

(c) \((S, P)\) and \((T, Q)\) are weakly compatible.

Then \( S, P, Q \) and \( T \) have a unique common fixed point in \( X \).

**Proof** In this part, we will show that \( \lim d(y_{n+1}, y_n, a) = 0 \).

Let \( x_0 \) be an arbitrary point in \( X \) and construct two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\begin{align*}
y_n &= Qx_{n+1} = Sx_n, \\
y_{n+1} &= Px_{n+2} = Tx_{n+1}.
\end{align*}
\]

From (b), we have

\[
d(y_{n+1}, y_n, a) = d(Sx_n, Tx_{n+1}, a) \leq c\lambda(x_n, x_{n+1}, a) \tag{3.1}
\]

where

\[
\begin{align*}
\lambda(x_n, x_{n+1}, a) &= \max\{d(Px_n, Qx_{n+1}, a), d(Px_n, Sx_n, a), d(Qx_{n+1}, Tx_{n+1}, a)\} \\
&= \max\{d(Tx_{n-1}, Sx_n, a), d(Tx_{n-1}, Sx_n, a), d(Sx_n, Tx_{n+1}, a)\} \\
&= \max\{d(Tx_{n-1}, Sx_n, a), d(Sx_n, Tx_{n+1}, a)\} \\
&= \max\{d(y_{n-1}, y_n, a), d(y_n, y_{n+1}, a)\}.
\end{align*}
\]

Assume \( \lambda(x_n, x_{n+1}, a) = d(y_n, y_{n+1}, a) \) and from (3.1) we have,

\[
d(y_n, y_{n+1}, a) < cd(y_n, y_{n+1}, a),
\]

which is impossible. Then we get \( \lambda(x_n, x_{n+1}, a) = d(y_{n-1}, y_n, a) \) also from (3.1) we get

\[
d(y_n, y_{n+1}, a) < cd(y_{n-1}, y_n, a) \tag{3.2}
\]

This implies that the sequence \( \{d(y_n, y_{n+1}, a)\} \) is decreasing and it must converge to \( r \geq 0 \). Therefore as \( n \to \infty \), from (3.2) we get \( r \leq cr \), this gives us that \( r = 0 \), then the result is obtained:

\[
\lim_{n \to \infty} d(y_{n+1}, y_n, a) = 0 \tag{3.3}
\]

Then we show that \( d(y_n, y_{n+1}, a) = 0 \).

From part 2 of Definition 2.5, we have \( d(x_m, x_m, x_{m-1}) = 0 \). Since \( \{d(x_n, x_{n+1}, a)\} \) is decreasing, we get \( d(x_n, x_{n+1}, a) = 0 \) from the assumption that \( d(x_{n-1}, x_n, a) = 0 \), then it is easy to get

\[
d(x_n, x_{n+1}, x_m) = 0, \quad \text{for all} \quad n+1 \geq m. \tag{3.4}
\]
For $0 \leq n+1 < m$, we get $m-1 \geq n+1$ and that is $m-2 \geq n$, from (3.4)
\[ d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0, \quad (3.5) \]

For (3.5) and triangular inequality, we have
\[ d(x_n, x_{n+1}, x_m) \leq s d(x_n, x_{n+1}, x_{m-1}) + s d(x_{n+1}, x_m, x_{m-1}) \]
\[ = d(x_n, x_{n+1}, x_{m-1}). \]

And since $d(x_n, x_{n+1}, x_{m+1}) = 0$, and from the inequality above,
\[ d(x_{n+1}, x_n, x_m) \leq s^{m-n-1} d(x_{n+1}, x_{n+1}, x_n) = 0, \quad \text{for all} \quad 0 \leq n+1 \leq m. \quad (3.6) \]

Now for all $i, j, k \in N$, now we consider the condition of $j > i$, from the above equation
\[ d(x_{j-1}, x_j, x_i) = d(x_k, x_{j-1}, x_j) = 0. \quad (3.7) \]

From (3.7) and triangular inequality, therefore
\[ d(x_i, x_k, x_j) \leq s [d(x_i, x_j, x_{j-1}) + d(x_j, x_{j-1}, x_k) + d(x_k, x_{j-1}, x_j)] \]
\[ \leq s \Lambda \]
\[ \leq s^{j-i} d(x_i, x_k, x_j) \]
\[ = 0. \]

In conclusion, the result below is gotten
\[ d(x_i, x_k, x_j) = 0, \quad \text{for all} \quad i, j, k \in N. \quad (3.8) \]

Now we prove that $\{y_n\}$ is a Cauchy sequence.

Suppose to the contrary, that is, $\{y_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$
for which we can find two subsequences $\{n_i\}$ and $\{m_i\}$ such that $i < m_i < n_i$ and
\[ d(y_{m_i}, y_n, a) \geq \varepsilon \quad \text{and} \quad d(y_{m_i}, y_{n-1}, a) < \varepsilon. \quad (3.9) \]

From the part 4 of Definition 2.5 and (3.8), we get
\[ d(y_{m_i}, y_{n-1}, a) \leq s [d(y_{m_i}, y_{m_{i+1}}, a) + d(y_{m_{i+1}}, y_n, a) + d(y_n, y_{m_{i+1}}, a)] \]
\[ \leq s d(y_{m_i}, y_{m_{i+1}}, a) + d(y_{m_i}, y_n, a). \]

Taking $i \to \infty$, from (3.3) and (3.9) we have
\[ \varepsilon \leq 1 \lim_{i \to \infty} d(y_{m_{i+1}}, y_{n_i}, a). \quad (3.10) \]

From (b), we get
\[ d(y_{n_i}, y_{m_{i+1}}, a) = d(Sx_{n_i}, Tx_{m_{i+1}}, a) \leq c \Lambda(x_{n_i}, y_{m_{i+1}}, a). \quad (3.11) \]

Since
\[ \lim_{n \to \infty} \Lambda(x_{n_i}, x_{m_{i+1}}, a) = \max \{ \lim_{n \to \infty} d(Px_{n_i}, Qx_{m_{i+1}}, a), \lim_{n \to \infty} d(Px_{n_i}, Sx_{m_{i+1}}, a), \]
\[ = \max \{ \lim_{n \to \infty} d(y_{n_i-1}, y_{m_i}, a), \lim_{n \to \infty} d(y_{n_i-1}, y_{n_i}, a), \lim_{n \to \infty} d(y_{m_i+1}, y_{m_i}, a) \} \]
\[ = \lim_{n \to \infty} d(y_{n_i-1}, y_{m_i}, a). \]

And by (3.11) we have
\[ \lim_{n \to \infty} d(y_{n_i}, y_{m_{i+1}}, a) \leq \lim_{n \to \infty} c \Lambda(y_{n_i}, y_{m_{i+1}}, a). \quad (3.12) \]

Again taking $i \to \infty$ by (3.9) and (3.12) we get
\[ \varepsilon \leq 1 \lim_{i \to \infty} d(y_{m_{i+1}}, y_{n_i}, a) \leq c \varepsilon < \frac{\varepsilon}{s}. \quad (3.13) \]
Which is a contraction. Therefore \( \{ y_n \} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, there exists a point \( z \in X \) such that \( n \to \infty, \{ y_n \} \to z \).

Thus \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} Q x_n = z \) and \( \lim_{n \to \infty} T x_{n+1} = \lim_{n \to \infty} P x_{n+2} = z \).

That is \( \lim_{n \to \infty} S x_n = \lim_{n \to \infty} Q x_n = \lim_{n \to \infty} T x_{n+1} = \lim_{n \to \infty} P x_{n+2} = z \). From \( T(X) \subseteq P(X) \) and \( P \) is a surjection, there exists a point \( u \) in \( X \) such that \( z = Pu \), then from (b), we get

\[
d(Su, z, a) \leq s[d(Su, Tx_{n+1}, a) + d(Tx_{n+1}, z, a) + d(Tx_{n+1}, Su, z)]
\]

\[
\leq s[c \lambda(u, x_{n+1}, a) + d(Tx_{n+1}, z, a) + d(Tx_{n+1}, Su, a)],
\]

where

\[
\lambda(u, x_{n+1}, a) = \max\{d(Pu, Qx_{n+1}, a), d(Pu, Su, a), d(Qx_{n+1}, Tx_{n+1}, a)\}
\]

\[
= \max\{d(z, Sx_n, a), d(z, Su, a), d(Sx_n, Tx_{n+1}, a)\}.
\]

We take \( n \to \infty \), we get

\[
\lambda(u, x_{n+1}, a) = \max\{d(z, z, a), d(z, Su, a), d(z, z, a)\} = d(z, Su, a).
\]

Therefore as \( n \to \infty \), \( d(Su, z, a) \leq sc(d(z, Su, a)) \).

Assume there exists \( a \in X \) such that \( d(Su, z, a) > 0 \) then we get \( \frac{1}{s} \leq c \) from the above inequality, which is contraction with \( c < \frac{1}{s} \). Thus \( Su = z \), furthermore \( Pu = Su = z \). So \( P \) and \( S \) have a coincidence point \( u \) in \( X \). Since \( P \) and \( S \) are weakly compatible, \( SPu = PSu \) that is \( Sz = Pz \).

From \( S(X) \subseteq Q(X) \) and \( Q \) is a surjection, there exists a point \( v \) in \( X \) such that \( z = Qv \), then from (b), we get

\[
d(Tv, z, a) \leq c \lambda(u, v, a),
\]

where

\[
\lambda(u, v, a) = \max\{d(Pu, Qv, a), d(Pu, Su, a), d(Qv, Tv, a)\}
\]

\[
= \max\{d(z, z, a), d(z, z, a), d(z, Tv, a)\}
\]

\[
= d(z, Tv, a).
\]

Then

\[
d(z, Tv, a) \leq cd(z, Tv, a).
\]

Assume \( d(z, Tv, a) > 0 \), then we have \( 1 \leq c \), which is contraction with \( c < \frac{1}{s} < 1 \).

Therefore \( Tv = Qv = z \). So \( Q \) and \( T \) have a coincidence point \( v \) in \( X \). Since \( Q \) and \( T \) are weakly compatible, \( QTv = TQv \) that is \( Qz = Tz \).

Now we prove that \( z \) is a fixed point of \( S \). By (b), we get

\[
d(Sz, z, a) = d(Sz, Tv, a) \leq c \lambda(z, v, a),
\]

where

\[
\lambda(z, v, a) = \max\{d(Pz, Qv, a), d(Pz, Sz, a), d(Qv, Tv, a)\}
\]

\[
= \max\{d(Sz, z, a), d(Sz, Sz, a), d(z, z, a)\}
\]

\[
= d(Sz, z, a).
\]

then we get

\[
d(Sz, z, a) \leq cd(Sz, z, a).
\]

Assume \( d(z, Tv, a) > 0 \), we get \( 1 \leq c \), which is a contraction. Thus \( Sz = Pz = z \).

Now we prove that \( z \) is a fixed point of \( T \). Then from (b), we get
\[ d(Tz, z, a) = d(Sz, Tz, a) \leq c \lambda(z, z, a), \]

where
\[
\lambda(z, z, a) = \max\{d(Pz, Qz, a), d(Pz, Sz, a), d(Qz, Tz, a)\}
= \max\{d(Tz, z, a), d(Tz, Tz, a), d(z, z, a)\}
= d(Tz, z, a).
\]

then we get
\[ d(z, Tz, a) \leq cd(z,Tv, a). \]

Assume \( d(z, Tz, a) > 0 \), we have \( 1 \leq c \), which is a contraction. Thus \( Tz = Qz = z \).

So we get \( z \) is a common fixed point of \( P, Q, S, T \). From (b), we get
\[ d(z, \omega, a) = d(Sz, Tz, a) \leq c \lambda d(z, \omega, a), \]

where
\[
\lambda(z, \omega, a) = \max\{d(Pz, Q\omega, a), d(Pz, Sz, a), d(Qz, T\omega, a)\}
= \max\{d(z, \omega, a), d(z, z, a), d(\omega, \omega, a)\}
= d(z, \omega, a).
\]

thus \( d(z, \omega, a) \leq c \lambda d(z, \omega, a). \)

Suppose that \( d(z, \omega, a) > 0 \), we get \( 1 \leq c \), which is a contraction. Thus \( z = \omega \), then
\( P, Q, S, T \) have a unique common fixed point \( z \in X \). \( \square \)

REFERENCES


