CONSTRUCTION OF MUTUALLY UNBIASED MAXIMALLY ENTANGLED BASES IN $C^3 \otimes C^{12}$ FROM $C^3 \otimes C^3$

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Abstract: In this paper, we first investigate two pairs of mutually unbiased maximally entangled bases in bipartite systems $C^3 \otimes C^3$. According to the mutually unbiased property of maximally entangled bases, we simplify the mutually unbiased problem to the choice of the unitary transformation matrix in C^3 and there are many unitary matrixes to be selected. Then we present more pairs maximally entangled bases in $C^3 \otimes C^{12}$ from that of $C^3 \otimes C^3$.

Keywords: Mutually unbiased bases; maximally entangled state; maximally entangled basis.

1. Introduction

Mutually unbiased bases play central roles in quantum kinematics [1], quantum state tomography [2-3] and in quantifying wave-particle duality in multipath interferometers [4]. Two orthonormal bases $B_1 = \{|\phi_i\rangle\}_{i=1}^{d}$ and $B_2 = \{|\psi_j\rangle\}_{j=1}^{d}$ of C^d are said to be mutually unbiased [5] if

$$\left|\langle\phi_i|\psi_j\rangle\right| = \frac{1}{\sqrt{d}}, \qquad i, j = 1, 2, \ldots, d$$

We also call that $B_1 = \{|\phi_i\rangle\}_{i=1}^{d}$ and $B_2 = \{|\psi_j\rangle\}_{j=1}^{d}$ are mutually unbiased bases(MUBs). Ever since the introduction of MUBs, considerable theoretical results with useful applications have been obtained. One main concern is about the maximal number of MUBs for given dimension d. It has been shown that the maximum number $N(d)$ of MUBs in C^d is no more than $d+1$ [3] and $N(d) = d+1$ if d is a prime power. While d is a composite number, $N(d)$ is still unknown.

In a bipartite system $C^d \otimes C^{d'}$ of composite dimension dd', there are different kinds of bases according to the entanglement of basis vectors such as product basis (PB) [6], unextendible product basis (UPB) [7], unextendible maximally entanged basis (UMEB) [8-12], maximally entangled basis (MEB) [13-19], entangled basis with Schmidt number k(EBk) [20-21].

In 2015, Tao et al. [13] studied mutually unbiased maximally entangled bases (MUMEBs) in $C^d \otimes C^{kd} (k \in Z^+)$ and established five MUMEBs in $C^2 \otimes C^4$ and three MUMEBs in $C^2 \otimes C^6$.

Zhang et al.[14] discussed the condition of two mutually unbiased maximally entangled bases and used special transforming unitary matrices of the bases in $\mathbb{C}^4$ to construct five MUMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^4$.

Zhang et al. [15] constructed MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}(d'=kd, k,l \in \mathbb{Z}^+)$ from MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}(d'=kd, k \in \mathbb{Z}^+)$. In 2016, Nan et.al. [16] presented a systematic way of constructing MUMEBs in bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{d^k}$. Luo et. Al .[17] constructed a new MEB in $\mathbb{C}^d \otimes \mathbb{C}^{kd}(k \in \mathbb{Z}^+)$ different from that in [13], then generalized MEBs into arbitrary bipartite systems $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ and discussed mutually unbiased property of two kinds MEBs. In 2017, Liu et al. [18] investigated $d-1$MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d (d \geq 3)$ by using any commutative ring R with d elements and generic character of $(R,+)$. In 2018, Xu [19] constructed $2(d-1)$ MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d (d \geq 3)$.

A pure state $|\psi\rangle$ is said to be a $d \otimes d'(d \leq d')$ maximally entangled state if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of subsystem A, there exists an orthonormal basis $\{|i_B\rangle\}$ of subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$. A set of states $\{|\varphi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} : i = 1,2,...,dd'\}$ is said to be an maximally entangled basis (MEB) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ if and only if (i) each $|\varphi_i\rangle (i=1,2,...,dd')$ is maximally entangled; (ii) $\langle \varphi_i | \psi_j \rangle = \delta_{ij}, i, j = 1,2,...,dd'$

In this paper, we first analyze MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ constructed in[15], then transfer the construction of MUMEB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ to the choice of transforming unitary matrices of the bases in $\mathbb{C}^3$. Using the same method, we present more pairs of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$ from that in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

2. Transformation matrix A in $\mathbb{C}^3$ for constructing MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$

In this section, we discuss the construction of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$. Zhang et.al [15]. have established a pair of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ as follows:

Set $\{|p\rangle\}_{p=0}^{2}$ be the computational basis in $\mathbb{C}^3$ and $\{|\varepsilon_j'\rangle\}_{j=0}^{2}$ be another orthonormal basis in $\mathbb{C}^3$ satisfying that

$$(|\varepsilon_0'\rangle, |\varepsilon_1'\rangle, |\varepsilon_2'\rangle) = A_0(|0'\rangle, |1'\rangle, |2'\rangle), \tag{1}$$

Where $e^{\frac{2\pi\sqrt{-1}}{3}}$ and $A_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$.

Let

$$|\mu_0\rangle = \frac{1}{\sqrt{3}}(|00'\rangle + |11'\rangle + |22'\rangle),$$
$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle|\varepsilon_0'\rangle + |1\rangle|\varepsilon_1'\rangle + |2\rangle|\varepsilon_2'\rangle), \tag{2}$$

then the pair of MUMEBs $\{|\varphi_{n,m}^{(0)}\rangle\}$ and $\{|\varphi_{n,m}^{(1)}\rangle\}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ constructed in [15] are as follows:

$$|\varphi_{n,m}^{(0)}\rangle = (U_{n,m} \otimes I_3)|\mu_0\rangle, \qquad n,m = 0,1,2, \tag{3}$$
$$|\varphi_{n,m}^{(1)}\rangle = (U_{n,m} \otimes I_3)|\psi_0\rangle, \qquad n,m = 0,1,2, \tag{4}$$

where $\oplus_3$ is modulus 3 addition and

$$U_{n,m} = \sum_{p=0}^{2} \omega^{np} |p \oplus_3 m\rangle\langle p|, \qquad n,m = 0,1,2. \tag{5}$$

Obviously, we can describe the second MEB $\{|\varphi_{n,m}^{(1)}\rangle\}$ in (4) as follows:

$$|\varphi_{n,m}^{(1)}\rangle = (U_{n,m} \otimes A)|\mu_0\rangle, \qquad n,m = 0,1,2. \tag{6}$$

Then there arose a question: are there any other unitary matrices like A_0 in (6) to construct new MEB mutually unbiased to $\{|\varphi_{n,m}^{(0)}\rangle\}$? To answer this question, we first assume that

$$|\varphi_{n,m}^{(2)}\rangle = (U_{n,m} \otimes A)|\mu_0\rangle, \qquad n,m = 0,1,2 \tag{7}$$

Note that $\{|\varphi_{n,m}^{(0)}\rangle\}$ and $\{|\varphi_{n,m}^{(2)}\rangle\}$ are mutually unbiased if and only if

$$\left|\left\langle \varphi_{n,m}^{(0)} \middle| \varphi_{n,m}^{(2)} \right\rangle\right| = \frac{1}{3}, \qquad n,n',m,m' = 0,1,2. \tag{8}$$

Denote $A = \left(a_{i,j}\right)$, then

$$\left|\left\langle \varphi_{n,m}^{(0)} \middle| \varphi_{n',m'}^{(2)} \right\rangle\right|$$
$$= \left|\left\langle \mu_0 \middle| U_{n,m}^{\dagger} U_{n,m} \otimes A \middle| \mu_0 \right\rangle\right|$$
$$= \left|\frac{1}{3}\left(\langle 00| + \langle 11| + \langle 22|\right)\left[(\sum_{p=0}^{2} \omega^{-np} |p\rangle\langle p \oplus_3 m|)(\sum_{p'=0}^{2} \omega^{n'p'} |p'\oplus_3 m'\rangle\langle p'|) \otimes A \right](|00\rangle + |11\rangle + |22\rangle)\right|,$$

where $n', n, m', m = 0,1,2$. According to the values of n', n, m' and m, then the condition (8) can be summarized as follows:

$$\begin{cases} |a_{11} + a_{22} + a_{33}| + |a_{11} + a_{22}\omega + a_{33}\omega^2| = |a_{11} + a_{22}\omega^2 + a_{33}\omega| = 1, \\ |a_{12} + a_{23} + a_{31}| + |a_{12} + a_{23}\omega + a_{31}\omega^2| = |a_{12} + a_{23}\omega^2 + a_{31}\omega| = 1, \\ |a_{13} + a_{21} + a_{32}| + |a_{13} + a_{21}\omega + a_{32}\omega^2| = |a_{13} + a_{21}\omega^2 + a_{32}\omega| = 1, \end{cases} \tag{10}$$

That is to say, the two MEBs $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$ and $\left\{\left|\varphi_{n,m}^{(2)}\right\rangle\right\}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are mutually unbiased if and only if (10) hold. thus we transfer the problem of construction of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ to the construction of unitary matrix A satisfying (10). It is easy to verify that the following three unitary matrices A_1 A_2 and A_3, different from A_0, satisfy the conditions of (10).

$$A_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}, \quad A_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} \omega & 1 & 1 \\ \omega & \omega^2 & \omega \\ \omega & \omega & \omega^2 \end{pmatrix},$$

Then using unitary matrices A_1 A_2 and A_3, we construct the following three more MEBs mutually unbiased to $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\left|\varphi_{n,m}^{(2)}\right\rangle = \left(U_{n,m} \otimes A_1\right)|\mu_0\rangle, \qquad n,m = 0,1,2 \tag{11}$$
$$\left|\varphi_{n,m}^{(3)}\right\rangle = \left(U_{n,m} \otimes A_2\right)|\mu_0\rangle, \qquad n,m = 0,1,2 \tag{12}$$
$$\left|\varphi_{n,m}^{(4)}\right\rangle = \left(U_{n,m} \otimes A_3\right)|\mu_0\rangle, \qquad n,m = 0,1,2 \tag{13}$$

In short, $\left\{\left|\varphi_{n,m}^{(1)}\right\rangle\right\}$ and $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$, $\left\{\left|\varphi_{n,m}^{(2)}\right\rangle\right\}$ and $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$, $\left\{\left|\varphi_{n,m}^{(3)}\right\rangle\right\}$ and $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$, $\left\{\left|\varphi_{n,m}^{(3)}\right\rangle\right\}$ and $\left\{\left|\varphi_{n,m}^{(0)}\right\rangle\right\}$ are three pairs of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

2. Transformation in $\mathbb{C}^{12}$ for constructing MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$

Based on one pair of MUMEBs $\{|\varphi_{n,m}^{(0)}\rangle\}$ and $\{|\varphi_{n,m}^{(1)}\rangle\}$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$, Zhang [15] have presented a pair of MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$ as follows:

Let $\{|0'\rangle, |1'\rangle, ..., |11'\rangle\}$ and $\{|v_0'\rangle, |v_1'\rangle, ..., |v_{11}'\rangle\}$ be two orthonormal bases in $\mathbb{C}^{12}$, and

$$(|v_0'\rangle, |v_1'\rangle, ..., |v_{11}'\rangle) = T_{00}(|0'\rangle, |1'\rangle, ..., |11'\rangle), \tag{14}$$

where

$$T_{00} = \begin{pmatrix} A_0 & A_0 & A_0 & A_0 \\ A_0 & -A_0 & A_0 & -A_0 \\ A_0 & A_0 & -A_0 & -A_0 \\ A_0 & -A_0 & -A_0 & A_0 \end{pmatrix}. \tag{15}$$

Then the two MUMEBs $\mathbb{C}^3 \otimes \mathbb{C}^{12}$ are constructed in [15] as follows:

$$|\phi_{n,m}^l\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} \omega^{np} |p \oplus m\rangle \otimes |(3l+p)'\rangle, \quad n, m = 0,1,2; \quad l = 0,1,2,3, \tag{16}$$

$$|\psi_{n,m}^l\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} \omega^{np} |p \oplus m\rangle \otimes T_{0,0} |(3l+p)'\rangle, \quad n, m = 0,1,2; \quad l = 0,1,2,3, \tag{17}$$

Obviously we can describe $T_{0,0}$ as follows,

$$T_{0,0} = B_0 \otimes B_0 \otimes A_0 \tag{18}$$

where $B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Then there arose a question: are there any other unitary matrices like $T_{0,0}$ in (18) to construct new MEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$ mutually unbiased to $\{|\phi_{n,m}^l\rangle\}$? To answer this question, we first assume that

$$|\xi_{n,m}^l\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} \omega^{np} |p \oplus m\rangle \otimes T_{s,t} |(3l+p)'\rangle, \quad n, m = 0,1,2; \quad l = 0,1,2,3, \tag{19}$$

Note that $\{|\phi_{n,m}^l\rangle\}$ and $\{|\xi_{n,m}^l\rangle\}$ are mutually unbiased if and only if

$$|\langle \phi_{n,m}^l | \xi_{n',m'}^{l'} \rangle| = \frac{1}{6}, \quad n, n', m, m' = 0,1,2; \quad l, l' = 0,1,2,3. \tag{20}$$

Denote $T_{s,t}=B_s\otimes B_s\otimes A_t$, where $A_t=\left(a_{gh}\right)_{3\times3}$ is 3×3 matrix, $B_s=\left(b_{ij}\right)_{2\times2}$ is a 2×2 matrix, then

$$T_{s,t}=\begin{pmatrix} b_{11}^2A_t & b_{11}b_{12}A_t & b_{11}b_{12}A_t & b_{12}^2A_t \\ b_{11}b_{21}A_t & b_{11}b_{22}A_t & b_{12}b_{21}A_t & b_{12}b_{22}A_t \\ b_{11}b_{21}A_t & b_{12}b_{21}A_t & b_{11}b_{22}A_t & b_{12}b_{22}A_t \\ b_{21}^2A_t & b_{21}b_{22}A_t & b_{21}b_{22}A_t & b_{22}^2A_t \end{pmatrix}. \tag{21}$$

Since

$$\left|\left\langle \phi_{n,m}^l \middle| \xi_{n',m'}^{l'} \right\rangle\right| = \frac{1}{3}\left|\sum_{p=0}^{2}\omega^{-np}\left\langle p\oplus m\right|\left\langle(3l+p)'\right|\sum_{q=0}^{2}\omega^{n'q}\left|q\oplus m'\right\rangle T_{s,t}\left|(3l+q)'\right\rangle\right|, \tag{22}$$

where $n,n',m,m'=0,1,2;\ \ l,l'=0,1,2,3.$ According the values of n,n',m and m', we can summarize the mutually unbiased condition (20) as follows:

$$\begin{cases} \left|b_{i,j}b_{i',j'}\right|\left|a_{11}+a_{22}+a_{33}\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{11}+a_{22}\omega+a_{33}\omega^2\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{11}+a_{22}\omega^2+a_{33}\omega\right| = \frac{1}{2}, \\ \left|b_{i,j}b_{i',j'}\right|\left|a_{12}+a_{23}+a_{31}\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{12}+a_{23}\omega+a_{31}\omega^2\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{12}+a_{23}\omega^2+a_{31}\omega\right| = \frac{1}{2}, \\ \left|b_{i,j}b_{i',j'}\right|\left|a_{13}+a_{21}+a_{32}\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{13}+a_{21}\omega+a_{32}\omega^2\right| = \left|b_{i,j}b_{i',j'}\right|\left|a_{13}+a_{21}\omega^2+a_{32}\omega\right| = \frac{1}{2}, \end{cases} \tag{23}$$

where $i,j,i',j'=1,2.$

Obviously, if $\left|b_{11}\right|=\left|b_{12}\right|=\left|b_{21}\right|=\left|b_{22}\right|=\frac{1}{\sqrt{2}}$, then (23) become (10), i.e., we can choose all the matrices A_t satisfy (10). That is to say, the two MEBs $\left\{\left|\phi_{n,m}^l\right\rangle\right\}$ and $\left\{\left|\xi_{n,m}^l\right\rangle\right\}$ in $\mathbb{C}^3\otimes\mathbb{C}^{12}$ are mutually unbiased if and only if (23) and (10) hold. Thus we transfer the problem of construction of MUMEBs in $\mathbb{C}^3\otimes\mathbb{C}^{12}$ to the construction of unitary matrix B_s satisfying (23) . We can easily find that the following unitary matrices B_1, B_2 and B_3 different from B_0 satisfying

$$B_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix},\quad B_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix},\quad B_3=\frac{1}{\sqrt{2}}\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}.$$

Moreover, A_0 can be replaced by A_1, A_2 and A_3 in (14), so we can construct the following 12 unitary matrices $T_{s,t}=B_s\otimes B_s\otimes A_t$, so as to construct 12 MEBs that are mutually unbiased to $\left\{\left|\phi_{n,m}^l\right\rangle\right\}$

$$T_{0,1} = B_0 \otimes B_0 \otimes A_1, \quad T_{1,1} = B_1 \otimes B_1 \otimes A_1,$$
$$T_{0,2} = B_0 \otimes B_0 \otimes A_2, \quad T_{1,2} = B_1 \otimes B_1 \otimes A_2,$$
$$T_{0,3} = B_0 \otimes B_0 \otimes A_3, \quad T_{1,3} = B_1 \otimes B_1 \otimes A_3,$$
$$T_{2,1} = B_2 \otimes B_2 \otimes A_1, \quad T_{3,1} = B_3 \otimes B_3 \otimes A_1,$$
$$T_{2,2} = B_2 \otimes B_2 \otimes A_2, \quad T_{3,2} = B_3 \otimes B_3 \otimes A_2,$$
$$T_{2,3} = B_2 \otimes B_2 \otimes A_3, \quad T_{3,3} = B_3 \otimes B_3 \otimes A_3,$$

3. Another transformation matrix in $\mathbb{C}^{12}$ for constructing MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$

Besides the above 12 forms, $T_{s,t}$ can choose other structure. Assume that $T_{s,t} = C_k \otimes A_t$ in (21), obviously, we can choose $C_k = B_k \otimes B_k$; where $k = 0,1,2,3$. Now we try to find other forms of C_k, where $C_k \neq B_k \otimes B_k$. Then there arose a question: are there other unitary matrices like $T_{0,0}$ in (15) to construct new MEB mutually unbiased to $\{|\phi^l_{n,m}\rangle\}$? To answer this question, we first assume that

$$|\gamma^l_{n,m}\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} \omega^{np} |p \oplus m\rangle \otimes T_{k,t} |(3l+p)'\rangle, \quad n,m = 0,1,2; \quad l = 0,1,2,3, \quad (24)$$

Note that $\{|\phi^l_{n,m}\rangle\}$ and $\{|\gamma^l_{n,m}\rangle\}$ are mutually unbiased if and only if

$$|\langle \phi^l_{n,m} | \gamma^{l'}_{n',m'}\rangle| = \frac{1}{6}, \quad n,n',m,m' = 0,1,2; \quad l,l' = 0,1,2,3. \quad (25)$$

Denote $T_{k,t} = C_k \otimes A_t$, where $A_t = (a_{gh})_{3\times3}$ is 3×3 matrix, $C_k = (c_{ij})_{4\times4}$ is a 4×4 matrix, which can not be divide into $B_s \otimes B_s$, then

$$T_{k,t} = \begin{pmatrix} c_{11}A_t & c_{12}A_t & c_{13}A_t & c_{14}A_t \\ c_{21}A_t & c_{22}A_t & c_{23}A_t & c_{24}A_t \\ c_{31}A_t & c_{32}A_t & c_{33}A_t & c_{34}A_t \\ c_{41}A_t & c_{42}A_t & c_{43}A_t & c_{44}A_t \end{pmatrix}. \quad (26)$$

then (28) means that

$$|\langle \phi^l_{n,m} | \gamma^{l'}_{n',m'}\rangle| = \frac{1}{3}\left|\sum_{p=0}^{2} \omega^{-np} \langle p \oplus m | \langle (3l+p)' | \sum_{q=0}^{2} \omega^{n'q} |q \oplus m'\rangle T_{k,t} |(3l+q)'\rangle\right|, \quad (27)$$

where $n,n',m,m' = 0,1,2; \quad l,l' = 0,1,2,3.$ According the values of n,n',m and m', we can summarize the mutually unbiased condition (28) as follows:

$$\begin{cases} |c_{i,j}||a_{11}+a_{22}+a_{33}| = |c_{i,j}||a_{11}+a_{22}\omega+a_{33}\omega^2| = |c_{i,j}||a_{11}+a_{22}\omega^2+a_{33}\omega| = \frac{1}{2}, \\ |c_{i,j}||a_{12}+a_{23}+a_{31}| = |c_{i,j}||a_{12}+a_{23}\omega+a_{31}\omega^2| = |c_{i,j}||a_{12}+a_{23}\omega^2+a_{31}\omega| = \frac{1}{2}, \\ |c_{i,j}||a_{13}+a_{21}+a_{32}| = |c_{i,j}||a_{13}+a_{21}\omega+a_{32}\omega^2| = |c_{i,j}||a_{13}+a_{21}\omega^2+a_{32}\omega| = \frac{1}{2}, \end{cases} \quad (28)$$

Thus, we can summarize the mutually unbiased condition (25) to the choice of the unitary matrices A_t which satisfy condition (10) and the unitary matrices C_k that satisfy the following condition:

$$|c_{ij}| = \frac{1}{2}. \quad i, j = 1,2,3,4. \quad (29)$$

That is to say, the two MEBs $\{|\phi_{n,m}^l\rangle\}$ and $\{|\gamma_{n,m}^l\rangle\}$ in $C^3 \otimes C^{12}$ are mutually unbiased if and only if (29) and (10) hold. Thus we transfer the problem of construction of MUMEBs in $C^3 \otimes C^{12}$ to the construction of unitary matrix C_k satisfying (29). It is easy to find the following unitary matrices C_k satisfying (29):

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ i & i & -i & -i \\ i & -i & i & -i \\ 1 & -1 & -1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i & -1 & i & 1 \\ -i & -1 & i & -1 \\ i & -1 & -i & -1 \\ i & 1 & i & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i & -i & -1 & -1 \\ 1 & 1 & -i & i \\ -i & i & -1 & -1 \\ -1 & -1 & -i & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 & i & -i & -1 \\ -1 & i & i & 1 \\ -i & 1 & 1 & i \\ -i & 1 & -1 & -i \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} i & 1 & i & -1 \\ 1 & i & -1 & i \\ i & -1 & i & 1 \\ 1 & -i & -1 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 1 & i & 1 \\ 1 & i & 1 & -i \\ -i & -1 & i & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -i & i & i & -i \\ i & -i & i & -i \end{pmatrix}.$$

Furthermore, A_t can be A_0, A_1, A_2 and A_3, so we can construct the following 24 unitary matrices $T_{k,t}$ to construct 24 MEBs that are mutually unbiased to $\{|\phi_{n,m}^l\rangle\}$

$$T_{k,t} = C_k \otimes A_t, \qquad t = 0,1,2,3; \;\; k = 0,1,\ldots,8 \quad (30)$$

4. Conclusion

In this paper, we first analyzed MUMEBs in $C^3 \otimes C^3$ constructed in [15], then transfer the construction of MUMEB in $C^3 \otimes C^3$ to the choice of transforming unitary matrices of the bases in C^3. Thus we have two unitary matrices to construct MUMEBs in $C^3 \otimes C^3$. we first divide the C^{12} into $C^2 \otimes C^2 \otimes C^3$, and we discuss the unitary matrices of C^2 on the base of the unitary matrices of C^3 to construct the MUMEBs in $C^3 \otimes C^3$. Then we present more pairs of

MUMEBs in $C^3 \otimes C^{12}$ according to the choice of transforming unitary matrices $T_{s,t}$. Using the same method, we divide the C^{12} into $C^4 \otimes C^3$, and we discuss the unitary matrices $T_{k,t}$ of the bases in C^{12} on the base of the unitary matrices of C^3. So we get more pairs MUMEBs in $C^3 \otimes C^{12}$ from $C^3 \otimes C^3$.

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