FOURIER COEFFICIENTS OF A CLASS OF ETA QUOTIENTS OF WEIGHT 22 WITH LEVEL 12

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ABSTRACT

Williams[18] and later Yao, Xia and Jin[15] discovered explicit formulas for the coefficients of the Fourier series expansions of a class of eta quotients. Williams expressed all coefficients of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$ and Yao, Xia and Jin, following the method of proof of Williams, expressed only even coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$.Here, we will express the even Fourier coefficients of 1456 eta quotients in terms of $\sigma_{21}(n), \sigma_{21}\left(\frac{n}{2}\right), \sigma_{21}\left(\frac{n}{3}\right), \sigma_{21}\left(\frac{n}{4}\right), \sigma_{21}\left(\frac{n}{6}\right)$ and $\sigma_{21}\left(\frac{n}{12}\right)$.

Keywords: Dedekind eta function; eta quotients; Fourier series.

INTRODUCTİON

The divisor function $\sigma_i(n)$ is defined for a positive integer i by

$$\sigma_i(n) := \begin{cases} \sum_{d \text{ positive integer}, d|n} d^i, \text{if } n \text{ is a positive integer} \\ 0 \text{ if n is not a positive integer} \end{cases}. \quad (1)$$

The Dedekind eta function is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (2)$$

where

$$q := e^{2\pi i z}, z \in H = \{x + iy : y > 0\} \quad (3)$$

and an eta quotient of level n is defined by

$$f(z) := \prod_{m|n} \eta\,(mz)^{a_m}, n, m \in \mathbb{N}, a_m \in Z. \quad (4)$$

It is interesting and important to determine explicit formulas for the Fourier coefficients for eta quotients since they are the building blocks of modular forms of level n and weight k. The book of Köhler [13] (Chapter 3, pg.39) describes such expansions by means of Hecke Theta series and develops algorithms for the determination of suitable eta quotients. One can find more information in [3], [6], [14], [16], [17]. I have determined the Fourier coefficients of the theta series associated to some quadratic forms, see [7], [8] [9], [10], [11], and [12],
Recently, Williams, see [18] discovered explicit formulas for the coefficients of Fourier series expansions of a class of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)} = 1 + \sum_{n=1}^{\infty} c\,(n)q^n,$$

where

$$c(n) = 2\sigma(n) - 3\sigma(n/2) + 4\sigma(n/4) + 9\sigma(n/6) - 36\sigma(n/12).$$

Then Yao, Xia and Jin [15] expressed the even Fourier coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^{25}(2z)\eta^4(3z)}{\eta^{12}(z)\eta^5(4z)\eta^3(6z)\eta(12z)} = 1 + \sum_{n=1}^{\infty} c\,(n)q^n,$$

where

$$c(2n) = 65\sigma_3(n) - 68\sigma_3(n/2) - 81\sigma_3(n/3) + 324\sigma_3(n/6).$$

Motivated by these two results, we can express the even Fourier coefficients of 1456 eta quotients in terms of $\sigma_{21}(n), \sigma_{21}\left(\frac{n}{2}\right), \sigma_{21}\left(\frac{n}{3}\right), \sigma_{21}\left(\frac{n}{4}\right), \sigma_{21}\left(\frac{n}{6}\right)$ and $\sigma_{21}\left(\frac{n}{12}\right)$, see Table 4. One example is as follows:

$$\frac{\eta^{52}(2z)}{\eta^8(4z)} = 1 + \sum_{n=1}^{\infty} c\,(n)q^n,$$

where

$$c(2n) = -\frac{2891\,710\,464}{94\,481\,414\,901\,157}\sigma_{21}\left(\frac{n}{3}\right) + \frac{2891\,710\,464}{94\,481\,414\,901\,157}\sigma_{21}\left(\frac{n}{6}\right).$$

We can also find 594 eta quotients of weight 22 such that

$$c(2n-1) = 0$$

and even coefficients can be expressed by simple formula. Now we can state our main Theorem:

Now let

$$f_1 := \sum_{n=0}^{\infty} f_1\,(n)q^n = \frac{\eta^{29}(4z)\eta^{23}(6z)\eta^{17}(12z)}{\eta^{25}(2z)},$$

$$f_2 = \sum_{n=0}^{\infty} f_2\,(n)q^n = \frac{\eta^{24}(4z)\eta^{28}(6z)\eta^{16}(12z)}{\eta^{24}(2z)},$$

$$f_3 = \sum_{n=0}^{\infty} f_3\,(n)q^n = \frac{\eta^{26}(4z)\eta^{14}(6z)\eta^{26}(12z)}{\eta^{22}(2z)},$$

$$f_4 = \sum_{n=0}^{\infty} f_4\,(n)q^n = \frac{\eta^{28}(4z)\eta^{12}(6z)\eta^{12}(12z)}{\eta^{8}(2z)},$$

$$f_5 = \sum_{n=0}^{\infty} f_5\,(n)q^n = \frac{\eta^{16}(4z)\eta^{24}(6z)\eta^{24}(12z)}{\eta^{20}(2z)},$$

$$f_6 = \sum_{n=0}^{\infty} f_6\,(n)q^n = \frac{\eta^{28}(4z)\eta^{24}(6z)\eta^{12}(12z)}{\eta^{20}(2z)},$$

$$f_7 = \sum_{n=0}^{\infty} f_7\,(n)q^n = \frac{\eta^{11}(4z)\eta^{29}(6z)\eta^{23}(12z)}{\eta^{19}(2z)},$$

$$f_8 = \sum_{n=0}^{\infty} f_8\,(n)q^n = \frac{\eta^{18}(4z)\eta^{22}(6z)\eta^{10}(12z)}{\eta^{6}(2z)},$$

$$f_9 = \sum_{n=0}^{\infty} f_9\,(n)q^n = \frac{\eta^{25}(4z)\eta^{3}(6z)\eta^{21}(12z)}{\eta^{5}(2z)},$$

$$f_{10} = \sum_{n=0}^{\infty} f_{10}\,(n)q^n = \frac{\eta^{22}(4z)\eta^{30}(12z)}{\eta^{2}(2z)\eta^{6}(6z)},$$

$$f_{11} = \sum_{n=0}^{\infty} f_{11}\,(n)q^n = \frac{\eta^{23}(6z)\eta^{29}(12z)}{\eta(2z)\eta^{7}(4z)},$$

$$f_{12} = \sum_{n=0}^{\infty} f_{12}\,(n)q^n = \frac{\eta^{17}(4z)\eta^{23}(6z)\eta^{5}(12z)}{\eta(2z)},$$

$$f_{13} = \sum_{n=0}^{\infty} f_{12}\,(n)q^n = \frac{\eta^{29}(4z)\eta^{23}(6z)}{\eta(2z)\eta^{7}(12z)},$$

$$f_{14} = \sum_{n=0}^{\infty} f_{14}\,(n)q^n = \frac{\eta^{28}(6z)\eta^{28}(12z)}{\eta^{12}(4z)},$$

$$f_{15} = \sum_{n=0}^{\infty} f_{15}\,(n)q^n = \eta^{28}(6z)\eta^{16}(12z),$$

$$f_{16} = \sum_{n=0}^{\infty} f_{16}\,(n)q^n = \frac{\eta^{19}(2z)\eta^{25}(4z)\eta^{21}(12z)}{\eta^{21}(6z)},$$

$$f_{17} = \sum_{n=0}^{\infty} f_{17}\,(n)q^n = \frac{\eta^{23}(2z)\eta^{17}(4z)\eta^{29}(12z)}{\eta^{25}(6z)},$$

$$f_{18} = \sum_{n=0}^{\infty} f_{18}\,(n)q^n = \frac{\eta^{40}(2z)\eta^{16}(4z)}{\eta^{12}(6z)},$$

$$f_{19} = \sum_{n=0}^{\infty} f_{19}(n)q^n = \frac{\eta^{40}(2z)\eta^{4}(4z)\eta^{12}(12z)}{\eta^{12}(6z)},$$

$$f_{20} = \sum_{n=0}^{\infty} f_{20}(n)q^n = \frac{\eta^{39}(2z)\eta^{31}(6z)}{\eta^{15}(4z)\eta^{11}(12z)},$$

$$f_{21} = \sum_{n=0}^{\infty} f_{21}(n)q^n = \frac{\eta^{30}(4z)\eta^{40}(6z)}{\eta^{24}(2z)\eta^{2}(12z)},$$

$$f_{22} = \sum_{n=0}^{\infty} f_{22}(n)q^n = \frac{\eta^{37}(4z)\eta^{21}(6z)\eta^{9}(12z)}{\eta^{23}(2z)},$$

$$f_{23} = \sum_{n=0}^{\infty} f_{23}(n)q^n = \frac{\eta^{32}(4z)\eta^{2}(6z)\eta^{32}(12z)}{\eta^{22}(2z)},$$

$$f_{24} = \sum_{n=0}^{\infty} f_{24}(n)q^n = \frac{\eta^{39}(4z)\eta^{7}(6z)\eta^{19}(12z)}{\eta^{21}(2z)},$$

$$f_{25} = \sum_{n=0}^{\infty} f_{25}(n)q^n = \frac{\eta^{19}(4z)\eta^{27}(6z)\eta^{15}(12z)}{\eta^{17}(2z)},$$

$$f_{26} = \sum_{n=0}^{\infty} f_{26}(n)q^n = \frac{\eta^{26}(4z)\eta^{32}(6z)\eta^{2}(12z)}{\eta^{16}(2z)},$$

$$f_{27} = \sum_{n=0}^{\infty} f_{27}(n)q^n = \frac{\eta^{21}(4z)\eta^{13}(6z)\eta^{25}(12z)}{\eta^{15}(2z)},$$

$$f_{28} = \sum_{n=0}^{\infty} f_{28}(n)q^n = \frac{\eta^{33}(6z)\eta^{33}(12z)}{\eta^{11}(2z)\eta^{11}(4z)},$$

$$f_{29} = \sum_{n=0}^{\infty} f_{29}(n)q^n = \frac{\eta^{32}(4z)\eta^{14}(6z)\eta^{8}(12z)}{\eta^{10}(2z)},$$

$$f_{30} = \sum_{n=0}^{\infty} f_{30}(n)q^n = \frac{\eta^{3}(4z)\eta^{19}(6z)\eta^{31}(12z)}{\eta^{9}(2z)},$$

$$f_{31} = \sum_{n=0}^{\infty} f_{31}(n)q^n = \frac{\eta^{39}(4z)\eta^{19}(12z)}{\eta^{9}(2z)\eta^{5}(6z)},$$

$$f_{32}=\sum_{n=0}^{\infty} f_{32}(n)q^n=\frac{\eta^{34}(4z)\eta^{24}(6z)}{\eta^{8}(2z)\eta^{6}(12z)},$$

$$f_{33}=\sum_{n=0}^{\infty} f_{33}(n)q^n=\frac{\eta^{37}(4z)\eta^{33}(12z)}{\eta^{11}(2z)\eta^{15}(6z)},$$

$$f_{34}=\sum_{n=0}^{\infty} f_{34}(n)q^n=\frac{\eta^{34}(6z)\eta^{28}(12z)}{\eta^{6}(2z)\eta^{12}(4z)},$$

$$f_{35}=\sum_{n=0}^{\infty} f_{35}(n)q^n=\frac{\eta^{24}(4z)\eta^{40}(12z)}{\eta^{6}(2z)\eta^{14}(6z)},$$

$$f_{36}=\sum_{n=0}^{\infty} f_{31}(n)q^n=\frac{\eta^{36}(4z)\eta^{10}(6z)\eta^{4}(12z)}{\eta^{6}(2z)},$$

$$f_{37}=\sum_{n=0}^{\infty} f_{32}(n)q^n=\frac{\eta^{39}(6z)\eta^{15}(12z)}{\eta^{5}(2z)\eta^{5}(4z)},$$

$$f_{38}=\sum_{n=0}^{\infty} f_{33}(n)q^n=\frac{\eta^{25}(6z)\eta^{25}(12z)}{\eta^{3}(2z)\eta^{3}(4z)},$$

$$f_{39}=\sum_{n=0}^{\infty} f_{34}(n)q^n=\frac{\eta^{2}(2z)\eta^{32}(4z)\eta^{32}(12z)}{\eta^{22}(6z)}.$$

Now we can state our main Theorem:

Theorem 1. Let $b_1, b_2, \cdots, b_5$ be non-negative integers satisfying

$$b_1+b_2+\cdots+b_5\leq 44. \tag{5}$$

Define the integers $a_1, a_2, a_3, a_4, a_6, a_{12}$ by

$$a_1:=-b_1+2b_2-2b_3-4b_4-b_5+44, \tag{6}$$

$$a_2:=3b_1+b_2+3b_3+10b_4+b_5-110, \tag{7}$$

$$a_3:=3b_1+2b_2+6b_3+4b_4+3b_5-132, \tag{8}$$

$$a_4:=-2b_1-b_2-b_3-4b_4+2b_5+44, \tag{9}$$

$$a_6:=-9b_1-7b_2-9b_3-10b_4-7b_5+330, \tag{10}$$

$$a_{12}:=6b_1+3b_2+3b_3+4b_4+2b_5-132. \tag{11}$$

Now define integers

$$k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10,}k_{11},$$
$$k_{12}, k_{13}, k_{14}, k_{15}, k_{16}, k_{17}, k_{18}, k_{19}, k_{20}, k_{21},$$

$$k_{22}, k_{23}, k_{24}, k_{25}, k_{26}, k_{27}, k_{28}, k_{29}, k_{30}, k_{31}, k_{32},$$
$$k_{33}, k_{34}, k_{35}, k_{36}, k_{37}, k_{38}, k_{39}, k_{40}, k_{41}, k_{42}, k_{43} \text{ and } k_{44}$$

by

$$\frac{1}{2^{b_1+b_5}} x^{b_1}(1-x)^{b_2}(1+x)^{b_3}(1+2x)^{b_4}(2+x)^{b_5} \tag{12}$$
$$= k_0 + k_1 x + k_2 x^2 + k_3 x + k_4 x^4 + k_5 x^5 + k_6 x^6 + k_7 x^7 + k_8 x^8$$
$$+k_9 x^9 + k_{10} x^{10} + k_{11} x^{11} + k_{12} x^{12} + k_{13} x^{13} + k_{14} x^{14} + k_{15} x^{15} \tag{13}$$
$$+k_{16} x^{16} + k_{17} x^{17} + k_{18} x^{18} + k_{19} x^{19} + k_{20} x^{20} + k_{21} x^{21} + k_{22} x^{22} \tag{14}$$
$$+k_{23} x^{23} + k_{24} x^{24} + k_{25} x^{25} + k_{26} x^{26} + k_{27} x^{27} + k_{28} x^{28} + k_{29} x^{29} \tag{15}$$
$$+k_{30} x^{30} + k_{31} x^{31} + k_{32} x^{32} + k_{33} x^{33} + k_{34} x^{34} + k_{35} x^{35} + k_{36} x^{36} \tag{16}$$
$$+k_{37} x^{37} + k_{38} x^{38} + k_{39} x^{39} + k_{40} x^{40} + k_{41} x^{41} + k_{42} x^{42} + k_{43} x^{43} + k_{44} x^{44}. \tag{17}$$

Define the rational numbers

$$c_1, c_2, c_3, c_4, c_6, c_{12}, r_1, r_2, \dots, r_{38}$$

and r_{39} as in www.bariskendirli.com.tr/weight22/Table 1. Here

$$\{f_1, \dots f_{39}\} \backslash \{f_1, f_2, f_3, f_5, f_7, f_{16}, f_{17}, f_{22}, f_{23}, f_{34}, f_{39}\} \in S_{22}(\Gamma_0(12)),$$
$$\{f_1, f_2, f_3, f_5, f_7, f_{16}, f_{17}, f_{22}, f_{23}, f_{34}, f_{39}\} \subset M_{22}(\Gamma_0(12)) \backslash S_{22}(\Gamma_0(12))$$

and

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) = \delta(b_1) + \sum_{n=1}^{\infty} c\,(n) q^n,$$

where for n∈ ℕ,

$$c(n) = c_1\sigma_{21}(n) + c_2\sigma_{21}\left(\frac{n}{2}\right) + c_3\sigma_{21}\left(\frac{n}{3}\right) + c_4\sigma_{21}\left(\frac{n}{4}\right) + c_6\sigma_{21}\left(\frac{n}{6}\right) + c_{12}\sigma_{21}\left(\frac{n}{12}\right)$$
$$+r_1 f_1(n) + \cdots + r_{39} f_{39}(n).$$

In particular,

$$c(2n) = c_1\sigma_{21}(2n) + c_2\sigma_{21}(n) + c_4\sigma_{21}\left(\frac{n}{2}\right) + (4194305c_3 + c_6)\sigma_{21}\left(\frac{n}{3}\right)$$
$$+(c_{12} - 4194\,304c_3)\sigma_{21}\left(\frac{n}{6}\right) + r_1 f_1(2n) + \cdots + r_{20} f_{20}(2n),$$

$$c(2n-1) = c_1\sigma_{21}(2n-1) + c_3\sigma_{21}\left(\frac{2n-1}{3}\right)$$
$$+r_{21} f_{21}(2n-1) + \cdots + r_{39} f_{39}(2n-1),$$

for n∈ ℕ.

It follows from

(6-11) that

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12} = 24b_1, \tag{18}$$

$$a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 44, \tag{19}$$

$$-\frac{a_1}{6} - \frac{a_2}{3} - \frac{a_3}{6} - \frac{2a_4}{3} - \frac{a_6}{3} - \frac{2a_{12}}{3} = -b_1 - b_5.$$

Now we will use $p-k$ parametrization of Alaca, Alaca and Williams, see [1]:

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \quad (20)$$

where the theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{-\infty}^{\infty} q^{n^2}.$$

Setting $x = p$ in (12), and multiplying both sides by k^{22},we obtain

$$\frac{k^{22}}{2^{b_1+b_5}} p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}$$

$$= (k_0 + k_1 p + k_2 p^2 + k_3 p^3 + k_4 p^4 + k_5 p^5 + k_6 p^6 + k_7 p^7 + k_8 p^8 + k_9 p^9 + k_{10} p^{10}$$
$$+ k_{11} p^{11} + k_{12} p^{12} + k_{13} p^{13} + k_{14} p^{14} + k_{15} p^{15} + k_{16} p^{16} + k_{17} p^{17} + k_{18} p^{18} + k_{19} p^{19}$$
$$+ k_{20} p^{20} + k_{21} p^{21} + k_{22} p^{22} + k_{23} p^{23} + k_{24} p^{24} + k_{25} p^{25} + k_{26} p^{26} + k_{27} p^{27} + k_{28} p^{28}$$
$$+ k_{29} p^{29} + k_{30} p^{30} + k_{31} p^{31} + k_{32} p^{32} + k_{33} p^{33} + k_{34} p^{34} + k_{35} p^{35} + k_{36} p^{36}$$
$$+ k_{37} p^{37} + k_{38} p^{38} + k_{39} p^{39} + k_{40} p^{40} + k_{41} x^{41} + k_{42} x^{42} + k_{43} x^{43} + k_{44} x^{44}) k^{22}.$$

Alaca, Alaca and Williams [2] have established the following representations in terms of p and k:

$$\eta(q) = 2^{-1/6} p^{1/24} (1-p)^{1/2} (1+p)^{1/6} (1+2p)^{1/8} (2+p)^{1/8} k^{1/2}, \quad (1)$$

$$\eta(q^2) = 2^{-1/3} p^{1/12} (1-p)^{1/4} (1+p)^{1/12} (1+2p)^{1/4} (2+p)^{1/4} k^{1/2}, \quad (2)$$

$$\eta(q^3) = 2^{-1/6} p^{1/8} (1-p)^{1/6} (1+p)^{1/2} (1+2p)^{1/24} (2+p)^{1/24} k^{1/2}, \quad (3)$$

$$\eta(q^4) = 2^{-2/3} p^{1/6} (1-p)^{1/8} (1+p)^{1/24} (1+2p)^{1/8} (2+p)^{1/2} k^{1/2}, \quad (4)$$

$$\eta(q^6) = 2^{-1/3} p^{1/4} (1-p)^{1/12} (1+p)^{1/4} (1+2p)^{1/12} (2+p)^{1/12} k^{1/2}, \quad (5)$$

$$\eta(q^{12}) = 2^{-2/3} p^{1/2} (1-p)^{1/24} (1+p)^{1/8} (1+2p)^{1/24} (2+p)^{1/6} k^{1/2}, \quad (6)$$

$$E_6(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$= (1 - 246p - 5532p^2 - 38614p^3 - 135369p^4 - 276084p^5$$
$$- 348024p^6 - 276084p^7 - 135369p^8 - 38614p^9 - 5532p^{10}$$
$$- 246p^{11} + p^{12}) k^6,$$

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5$$
$$+ 964p^6 + 124p^7 + p^8) k^4.$$

Therefore, since

$$E_{22}(q)=\frac{20500}{77683}E_6^3(q)E_4(q)+\frac{57183}{77683}E_6(q)E_4^4(q),$$

We can also similarly determine $f_1,\cdots,f_{38}$ and f_{39} in terms of p and k as in www.bariskendirli.com.tr/weight22/Table 2.

We see that

$$\{f_1,\ldots f_{39}\}\backslash\{f_1,f_2,f_3,f_5,f_7,f_{16},f_{17},f_{22},f_{23},f_{34},f_{39}\}\in S_{22}\big(\Gamma_0(12)\big),$$
$$\{f_1,f_2,f_3,f_5,f_7,f_{16},f_{17},f_{22},f_{23},f_{34},f_{39}\}\subset M_{22}\big(\Gamma_0(12)\big)\backslash S_{22}\big(\Gamma_0(12)\big)$$

by [4]. Now

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z)$$
$$=q^{b_1}\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{3n})^{a_3}(1-q^{4n})^{a_4}(1-q^{6n})^{a_6}(1-q^{12n})^{a_{12}}$$
$$=2^{-\frac{a_1}{6}-\frac{a_2}{3}-\frac{a_3}{6}-\frac{2a_4}{3}-\frac{a_6}{3}-\frac{2a_{12}}{3}}p^{\frac{a_1}{24}+\frac{a_2}{12}+\frac{a_3}{8}+\frac{a_4}{6}+\frac{a_6}{4}+\frac{a_{12}}{2}}(1-p)^{\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{6}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}$$
$$(1+p)^{\frac{a_1}{6}+\frac{a_2}{12}+\frac{a_3}{2}+\frac{a_4}{24}+\frac{a_6}{4}+\frac{a_{12}}{8}}(1+2p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}(2+p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{2}+\frac{a_6}{12}+\frac{a_{12}}{6}}$$
$$k^{\frac{a_1+a_2+a_3+a_4+a_6+a_{12}}{2}}=\frac{k^{22}}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}$$

$$\begin{aligned}&=k^{20}(k_0+k_1p+k_2p^2+k_3p^3+k_4p^4+k_5p^5+k_6p^6\\&+k_7p^7+k_8p^8+k_9p^9+k_{10}p^{10}+k_{11}p^{11}\\&+k_{12}p^{12}+k_{13}p^{13}+k_{14}p^{14}+k_{15}p^{15}+k_{16}p^{16}\\&k_{17}p^{17}+k_{18}p^{18}+k_{19}p^{19}+k_{20}p^{20}+k_{21}p^{21}+k_{22}p^{22}\\&+k_{23}p^{23}+k_{24}p^{24}+k_{25}p^{25}+k_{26}p^{26}+k_{27}p^{27}+k_{28}p^{28}\\&+k_{29}p^{29}+k_{30}p^{30}+k_{31}p^{31}+k_{32}p^{32}+k_{33}p^{33}+k_{34}p^{34}\\&+k_{35}x^{35}+k_{36}x^{36}+k_{37}x^{37}+k_{38}x^{38}+k_{39}x^{39}+k_{40}p^{40}\\&+k_{41}x^{41}+k_{42}x^{42}+k_{43}x^{43}+k_{44}p^{44})\end{aligned}$$

$$=\frac{77683}{552}c_1\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^n\right)+\frac{77683}{552}c_2\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^{2n}\right)$$
$$+\frac{77683}{552}c_3\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^{3n}\right)+\frac{77683}{552}c_4\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^{4n}\right)$$
$$+\frac{77683}{552}c_6\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^{6n}\right)+\frac{77683}{552}c_{12}\left(1+\frac{552}{77683}\sum_{n=1}^{\infty}\sigma_{21}(n)q^{12n}\right)$$
$$+r_1q^{17}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{29}(1-q^{6n})^{23}(1-q^{12n})^{17}}{(1-q^{2n})^{25}}$$
$$+r_2q^{17}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{24}(1-q^{6n})^{28}(1-q^{12})^{16}}{(1-q^{2n})^{24}}$$
$$+r_3q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{26}(1-q^{6n})^{14}(1-q^{12n})^{26}}{(1-q^{2n})^{22}}$$
$$+r_4q^{13}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{28}(1-q^{6n})^{12}(1-q^{12n})^{12}}{(1-q^{2n})^{8}}$$

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$$+r_5q^{19}\prod_{n=1}^{\infty}\frac{((1-q^{4n})^{16}(1-q^{6n})^{24}(1-q^{12n})^{24}}{(1-q^{2n})^{20}}$$
$$+r_6q^{15}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{28}(1-q^{6n})^{24}(1-q^{12n})^{12}}{(1-q^{2n})^{20}}$$
$$+r_7q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{11}(1-q^{6n})^{29}(1-q^{12n})^{23}}{(1-q^{2n})^{19}}$$
$$+r_8q^{13}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{18}(1-q^{6n})^{22}(1-q^{12n})^{10}}{(1-q^{2n})^{6}}$$
$$+r_9q^{15}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{25}(1-q^{6n})^{3}(1-q^{12n})^{21}}{(1-q^{2n})^{5}}$$
$$+r_{10}q^{17}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{22}(1-q^{12n})^{30}}{(1-q^{2n})^{2}(1-q^{6n})^{6}}$$

$$+r_{11}q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{23}(1-q^{12n})^{29}}{(1-q^{2n})(1-q^{4n})^{7}}$$
$$+r_{12}q^{11}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{17}(1-q^{6n})^{23}(1-q^{12n})^{5}}{(1-q^{2n})}$$
$$+r_{13}q^{7}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{29}(1-q^{6n})^{23}}{(1-q^{2n})(1-q^{12n})^{7}}$$
$$+r_{14}q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{28}(1-q^{12n})^{28}}{(1-q^{4n})^{12}}$$
$$+r_{15}q^{15}\prod_{n=1}^{\infty}(1-q^{6n})^{28}(1-q^{12n})^{16}$$
$$+r_{16}q^{11}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{19}(1-q^{4n})^{25}(1-q^{12n})^{21}}{(1-q^{6n})^{21}}$$
$$+r_{17}q^{13}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{23}(1-q^{4n})^{17}(1-q^{12n})^{29}}{(1-q^{6n})^{25}}$$
$$+r_{18}q^{3}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{40}(1-q^{4n})^{16}}{(1-q^{6n})^{12}}$$
$$+r_{19}q^{7}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{40}(1-q^{4n})^{4}(1-q^{12n})^{12}}{(1-q^{6n})^{12}}$$
$$+r_{20}q^{3}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{39}(1-q^{6n})^{31}}{(1-q^{4n})^{15}(1-q^{12n})^{11}}$$
$$+r_{21}q^{12}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{30}(1-q^{6n})^{40}}{(1-q^{2n})^{24}(1-q^{12n})^{2}}$$

$$+r_{22}q^{14}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{37}(1-q^{6n})^{21}(1-q^{12n})^{9}}{(1-q^{2n})^{23}}$$
$$+r_{23}q^{20}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{32}(1-q^{6n})^{2}(1-q^{12n})^{32}}{(1-q^{2n})^{22}}$$
$$+r_{24}q^{16}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{39}(1-q^{6n})^{7}(1-q^{12n})^{19}}{(1-q^{2n})^{21}}$$
$$+r_{25}q^{16}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{19}(1-q^{6n})^{27}(1-q^{12n})^{15}}{(1-q^{2n})^{17}}$$

$$+r_{26}q^{12}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{26}(1-q^{6n})^{32}(1-q^{12n})^{2}}{(1-q^{2n})^{16}}$$
$$+r_{27}q^{18}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{21}(1-q^{6n})^{13}(1-q^{12n})^{25}}{(1-q^{2n})^{15}}$$
$$+r_{28}q^{22}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{33}(1-q^{12n})^{33}}{(1-q^{2n})^{11}(1-q^{4n})^{11}}$$
$$+r_{29}q^{12}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{32}(1-q^{6n})^{14}(1-q^{12n})^{8}}{(1-q^{2n})^{10}}$$
$$+r_{30}q^{20}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{3}(1-q^{6n})^{19}(1-q^{12n})^{31}}{(1-q^{2n})^{9}}$$
$$+r_{31}q^{14}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{39}(1-q^{12n})^{19}}{(1-q^{2n})^{9}(1-q^{6n})^{5}}$$
$$+r_{32}q^{8}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{34}(1-q^{6n})^{24}}{(1-q^{2n})^{8}(1-q^{12n})^{6}}$$
$$+r_{33}q^{18}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{37}(1-q^{12n})^{33}}{(1-q^{2n})^{11}(1-q^{6n})^{15}}$$
$$+r_{34}q^{20}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{34}(1-q^{12n})^{28}}{(1-q^{2n})^{6}(1-q^{4n})^{12}}$$
$$+r_{35}q^{20}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{24}(1-q^{12n})^{40}}{(1-q^{2n})^{6}(1-q^{6n})^{14}}$$

$$+r_{36}q^{10}\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{36}(1-q^{6n})^{10}(1-q^{12n})^{4}}{(1-q^{2n})^{6}}$$
$$+r_{37}q^{16}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{39}(1-q^{12n})^{15}}{(1-q^{2n})^{5}(1-q^{4n})^{5}}$$
$$+r_{38}q^{18}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{25}(1-q^{12n})^{25}}{(1-q^{2n})^{3}(1-q^{4n})^{3}}$$

$$+r_{39}q^{16}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{2}(1-q^{4n})^{32}(1-q^{12n})^{32}}{(1-q^{6n})^{22}}$$

$$=\delta(b_1)+\sum_{n=1}^{\infty}(c_1\sigma_{21}(n)+c_2\sigma_{21}\left(\frac{n}{2}\right)+c_3\sigma_{21}\left(\frac{n}{3}\right)+c_4\sigma_{21}\left(\frac{n}{4}\right)$$
$$+c_6\sigma_{21}\left(\frac{n}{6}\right)+c_{12}\sigma_{21}\left(\frac{n}{12}\right))+r_1f_1(n)+\ldots.+r_{39}f_{39}(n),$$

where

$$\delta(b_1)=\begin{cases}0 \text{ if } b_1\neq 0\\ 1 \text{ if } b_1=0\end{cases}.$$

So

$$c(n)=(c_1\sigma_{21}(n)+c_2\sigma_{21}\left(\frac{n}{2}\right)+c_3\sigma_{21}\left(\frac{n}{3}\right)+c_4\sigma_{21}\left(\frac{n}{4}\right)$$
$$+c_6\sigma_{21}\left(\frac{n}{6}\right)+c_{12}\sigma_{21}\left(\frac{n}{12}\right))+r_1f_1(n)+\ldots.+r_{39}f_{39}(n).$$

Therefore, for $n=1,2,\ldots$,

$$c(2n)=c_1\sigma_{21}(2n)+c_2\sigma_{21}(n)+c_4\sigma_{21}\left(\frac{n}{2}\right)+(1048\,577c_3+c_6)\sigma_{21}\left(\frac{n}{3}\right)$$
$$+(c_{12}-1048\,576c_3)\sigma_{21}\left(\frac{n}{6}\right)+r_1f_1(2n)+\cdots+r_{20}f_{20}(2n),$$

$$c(2n-1)=c_1\sigma_{15}(2n-1)+c_3\sigma_{15}\left(\frac{2n-1}{3}\right)$$
$$+r_{21}f_{21}(2n-1)+\cdots+r_{39}f_{39}(2n-1),$$

since it is easy to see that

$$\sigma_k\left(\frac{2n}{3}\right)=(2^k+1)\sigma_k\left(\frac{n}{3}\right)-2^k\sigma_k\left(\frac{n}{6}\right)$$

hence,

$$\sigma_{21}\left(\frac{2n}{3}\right)=1048\,577\sigma_{21}\left(\frac{n}{3}\right)-1048\,576\sigma_{21}\left(\frac{n}{6}\right),$$

and, for $n=1,2,\ldots$,

$$f_1(2n)=\cdots=f_{20}(2n)=0,$$

$$f_{21}(2n-1)=\cdots=f_{39}(2n-1)=0.$$

Remark 1. We have found 1456 eta quotients, see Table 4 for 300 of them, such that, for $n=1,2,\cdots$,

$$c(2n)=c_1\sigma_{21}(2n)+c_2\sigma_{21}(n)+c_4\sigma_{21}\left(\frac{n}{2}\right)+(1048\,577c_3+c_6)\sigma_{21}\left(\frac{n}{3}\right)$$
$$+(c_{12}-1048\,576c_3)\sigma_{21}\left(\frac{n}{6}\right)$$
$$c(2n-1)=c_1\sigma_{21}(2n-1)+c_3\sigma_{21}\left(\frac{2n-1}{3}\right)+r_1f_1(2n-1)+\cdots+r_{20}f_{20}(2n-1).$$

and 594 eta quotients, such that for $n=1,2,\cdots$,

$$c(2n)=c_1\sigma_{21}(2n)+c_2\sigma_{21}(n)+c_4\sigma_{21}\left(\frac{n}{2}\right)+c_6\sigma_{21}\left(\frac{n}{3}\right)$$
$$+c_{12}\sigma_{21}\left(\frac{n}{6}\right)+r_{21}f_{21}(2n)+\cdots+r_{39}f_{39}(2n),$$
$$c(2n-1)=0.$$

Remark 2. If f is an eta quotient, then the coefficients of $\frac{1}{2}(f(q)+f(-q))$ are exactly the even coefficients of f. In particular, it means that we have obtained all coefficients of some sum of 1456 eta quotients.
Remark 3. $S_{22}(\Gamma_0(12))$

is 39 dimensional, $M_{22}(\Gamma_0(12))$

is 45 dimensional,see [5] (Chapter 3, pg.87 and Chapter 5, pg.197), and generated by
$\Delta_{1,22},\Delta_{1,22}(2z),\Delta_{1,22}(3z),\Delta_{1,22}(4z),\Delta_{1,22}(6z),\Delta_{1,22}(12z),$
$\Delta_{2,22,1},\Delta_{2,22,1}(2z),\Delta_{2,22,1}(3z),\Delta_{2,22,1}(6z),$
$\Delta_{2,22,2},\Delta_{2,22,2}(2z),\Delta_{2,22,2}(3z),\Delta_{2,22,2}(6z),$
$\Delta_{3,22,1},\Delta_{3,22,1}(2z),\Delta_{3,22,1}(4z),$
$\Delta_{3,22,2},\Delta_{3,22,2}(2z),\Delta_{3,22,2}(4z),$
$\Delta_{3,22,3},\Delta_{3,22,3}(2z),\Delta_{3,22,3}(4z),$
$\Delta_{3,22,4}$(conjugate of $\Delta_{3,22,3}$ by $x^2-666x+2464992$), $\Delta_{3,22,4}(2z),\Delta_{3,22,4}(4z),$
$\Delta_{4,22,1},\Delta_{4,22,1}(3z),$
$\Delta_{4,22,2}$(conjugate of $\Delta_{4,22,1}$ by $x^2-65640x-14536815984$), $\Delta_{4,22,2}(3z),$
$\Delta_{6,22,1},\Delta_{6,22,1}(2z),$
$\Delta_{6,22,2},\Delta_{6,22,2}(2z),$
$\Delta_{6,22,3},\Delta_{6,22,3}(2z),$
$\Delta_{12,22,1},\Delta_{12,22,2},\Delta_{12,22,3}$(conjugate of $\Delta_{12,22,2}$ by $x^2-28827900x-453753148117500$)
where $\Delta_{1,22}$ is the unique newform in $S_{22}(\Gamma_0(1))$

; $\Delta_{2,22,1}$, $\Delta_{2,22,2}$ are the unique newforms in $S_{22}(\Gamma_0(2))$

; $\Delta_{3,22,1},\Delta_{3,22,2},\Delta_{3,22,3}$ and $\Delta_{3,22,4}$ are the unique newforms in $S_{22}(\Gamma_0(3))$

, $\Delta_{4,22,1},\Delta_{4,22,2}$ are the unique newforms in $S_{22}(\Gamma_0(4))$

, $\Delta_{6,18,1},\Delta_{6,18,2}$ and $\Delta_{6,18,3}$ are the unique newforms in $S_{22}(\Gamma_0(6))$, and $\Delta_{12,22,1},\Delta_{12,22,2}$ and $\Delta_{12,22,3}$ are the unique newforms in $S_{22}(\Gamma_0(12))$

. By taking t as a root of $x^2-666x+2464992$, s as the root of $x^2-65640x-14536815984$ and u as the root of $x^2-28827900x-453753148117500$, we see that $f_1,\cdots,f_{39}$ as linear combinations in www.bariskendirli.com.tr/weight22/Table 3. We also see that the even coefficients of the newforms and their scalings by divisors of 12 can be obtained by $\sigma_{21}(n),\sigma_{21}\left(\frac{n}{2}\right),\sigma_{21}\left(\frac{n}{3}\right),\sigma_{21}\left(\frac{n}{4}\right),\sigma_{21}\left(\frac{n}{6}\right)$ and $\sigma_{21}\left(\frac{n}{12}\right)$.

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