NEW NEWTON-TYPE METHOD WITH $(k+2)$-ORDER CONVERGENCE FOR FINDING SIMPLE ROOT OF A POLYNOMIAL EQUATION

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ABSTRACT

The objective of this paper is to define a new Newton-type method for finding simple root of a polynomial. It is proved that the new one-point method has the convergence order of $(k+2)$ requiring n function evaluations per iteration, where k is the number of terms in the generating series. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve maximum convergence order 2^{n-1}, but the new method produces convergence order of $(k+2)$, which is better than the expected maximum convergence order of eight. Therefore, we show that the conjecture fails for a particular set of polynomial equations. We will demonstrate that the new method is very simple to construct.

Keywords: Newton-type method; Polynomial equation; Kung-Traub's conjecture; Efficiency index; Optimal order of convergence.

Subject Classifications: AMS (MOS): 65H05.

INTRODUCTION

In this paper, we present a new one-point $(k+2)$-order iterative method to find a simple root of a polynomial equation. Let $p(x_n) = a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + \cdots + a_nx_n^q$ be a polynomial of degree q and a_i are real constants. Finding zeros of a polynomial has been interest in theoretical and many areas of applied mathematics. It is well established that many higher order multi-point variants of the Newton-type method have been developed based on the Kung and Traub conjecture [4]. Here we present a new iterative method which has a better efficiency index than the classical Newton method [3,5,6,7,10]. This paper is actually a continuation form the previous studies [8,9], hence the new one-point method is applied to higher order polynomial equation.

For the purpose of this paper, we construct a new Newton-type iterative method of $(k+2)$-order for finding simple root of polynomial equations. The new one-point method presented in this paper only uses n evaluations of the function per iteration. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. In fact, we have obtained a higher order of convergence than the maximum order of convergence suggested by Kung and Traub conjecture [4]. We demonstrate that the Kung and Traub conjecture fails for a particular case.

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Preliminaries

In order to establish the order of convergence of the iterative method [3,5,6,10], some of the definitions are stated:

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence *p* is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = \lambda \neq 0 \tag{1}$$

where $p \in \mathbb{R}^{+}$ and λ is the asymptotic error constant (*AEC*). Let $e_n = x_n - \alpha$ be the error in the *n*th iteration, then the relation

$$e_{n+1} = \zeta e_n^p + \mathrm{O}\left(e_n^{p+1}\right), \tag{2}$$

is the error equation. If the error equation exists, then *p* is the order of convergence of the iterative method, [3,5,6,10].

Definition 2 Let *n* be the number of function evaluations of the iterative method. The efficiency of the iterative method is measured by the concept of efficiency index and defined as

$$E(n,p) = \sqrt[n]{p} \tag{3}$$

where *p* is the order of convergence of the method, [6].

Definition 3 (Kung and Traub conjecture) Let $x_{n+1} = g(x_n)$ define as an iterative function without memory with *n*-evaluations. Then

$$p(g) \leq p_{opt} = 2^{n-1}, \tag{4}$$

where p_{opt} is the maximum order, [4].

Convergence Analysis

In this section we define a new class of one-point $(k+2)$-order method for finding simple root of a polynomial equation. In fact, the new iterative method is an extension of the Thukral's quadratic and cubic methods, given in [8,9]. Hence, we will demonstrate that the new one-point method can be constructed to find a simple root of any order of polynomial equation. The order of convergence the new iterative method is determined by the *k*, that is number of terms in the generating series, hence depending on *k* we can construct any desired order of convergence.

The Method

The new one-point $(k+2)$-order Newton-type method is expressed by

$$x_{n+1} = x_n - rH_k\left(r, t_1, t_2, t_3, \cdots t_p\right) \tag{5}$$

where

$$H_k\left(r, t_1, t_2, t_3, \cdots t_p\right) = 1 + \sum_{i=1}^{k} AEC\left(r, t_1, t_2, t_3, \cdots t_p; i\right) r^i \tag{6}$$

$$\left.\begin{aligned} r &= \frac{f(x_n)}{f'(x_n)}, \quad t_1 = \frac{f''(x_n)}{2!\,f'(x_n)}, \quad t_2 = \frac{f'''(x_n)}{3!\,f'(x_n)}, \\ t_3 &= \frac{f^{(iv)}(x_n)}{4!\,f'(x_n)}, \quad t_4 = \frac{f^{(v)}(x_n)}{5!\,f'(x_n)}, \quad t_5 = \frac{f^{(vi)}(x_n)}{6!\,f'(x_n)}, \\ t_6 &= \frac{f^{(vii)}(x_n)}{7!\,f'(x_n)}, \quad t_7 = \frac{f^{(viii)}(x_n)}{8!\,f'(x_n)}, \quad t_8 = \frac{f^{(ix)}(x_n)}{9!\,f'(x_n)}, \cdots \end{aligned}\right\} \tag{7}$$

where x_0 is the initial guess and provided that denominators of (7) are not equal to zero. Now, we shall verify the convergence property of the new one-point $(k+2)$-order iterative method (5).

Theorem 1

Let $\alpha \in D$ be a simple zero of a sufficiently smooth function $f : D \subset \Re \to \Re$ for an open interval D. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new one-point iterative method defined by (5) is $(k+2)$.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha)=0$ and $f'(\alpha)\neq 0$, and the error is expressed as $e = x-\alpha$.

The Taylor series expansion and taking into account $f(\alpha)=0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + \cdots\right). \tag{8}$$

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + \cdots\right). \tag{9}$$

$$f''(x_n) = f'(\alpha)\left(2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + 42c_7e_n^5 + 56c_8e_n^6 + \cdots\right). \tag{10}$$

$$f'''(x_n) = f'(\alpha)\left(6c_3 + 24c_4e_n + 60c_5e_n^2 + 120c_6e_n^3 + 210c_7e_n^4 + 336c_8e_n^5 + \cdots\right). \tag{11}$$

$$f^{(iv)}(x_n) = f'(\alpha)\left(24c_4 + 120c_5e_n + 360c_6e_n^2 + 840c_7e_n^3 + 1680c_8e_n^4 + \cdots\right). \tag{12}$$

$$f^{(v)}(x_n) = f'(\alpha)\left(120c_5 + 720c_6e_n + 2520c_7e_n^2 + 6720c_8e_n^3 + \cdots\right). \tag{13}$$

$$f^{(vi)}(x_n) = f'(\alpha)\left(720c_6 + 2520c_7e_n + 20160c_8e_n^2 + \cdots\right). \tag{14}$$

$$f^{(vii)}(x_n) = f'(\alpha)\left(2520c_7 + 40320c_8e_n + \cdots\right). \tag{15}$$

$\vdots$

where

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$$c_2 = \frac{f''(\alpha)}{f'(\alpha)}, \quad c_3 = \frac{f'''(\alpha)}{f'(\alpha)}, \quad c_4 = \frac{f^{(iv)}(\alpha)}{f'(\alpha)}, \quad c_5 = \frac{f^{(v)}(\alpha)}{f'(\alpha)}, \quad c_6 = \frac{f^{(vi)}(\alpha)}{f'(\alpha)}, \quad \cdots$$
(16)

Dividing (8) by (9), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \cdots.$$
(17)

and

$$\left(\frac{f''(x_n)}{f'(x_n)}\right) = 2c_2 - 2\left(2c_2^2 - 3c_3\right)e_n + 2c_2\left(4c_2^2 - 9c_3 + 6c_4\right)e_n^2 \cdots.$$
(18)

$$\left(\frac{f'''(x_n)}{f'(x_n)}\right) = 6c_3 + 12\left(2c_4 - c_2 c_3\right)e_n + \cdots.$$
(19)

$$\left(\frac{f^{(iv)}(x_n)}{f'(x_n)}\right) = 24c_4 + 24\left(5c_5 - 2c_2 c_4\right)e_n + \cdots.$$
(20)

$$\left(\frac{f^{(iv)}(x_n)}{f'(x_n)}\right) = 120c_5 + 240\left(3c_6 - c_2 c_5\right)e_n + \cdots.$$
(21)

$$\left(\frac{f^{(vi)}(x_n)}{f'(x_n)}\right) = 720c_6 + 720\left(7c_7 - 2c_2 c_6\right)e_n + \cdots.$$
(22)

$$\left(\frac{f^{(vii)}(x_n)}{f'(x_n)}\right) = 5040c_7 + \left(40320c_8 - 10080c_2 c_7\right)e_n + \cdots.$$
(23)

$$\left(\frac{f^{(viii)}(x_n)}{f'(x_n)}\right) = 40320c_8 + \left(362880c_9 - 80640c_2 c_8\right)e_n + \cdots.$$
(24)

$\vdots$

Substituting (19) in (1), we obtain

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} = c_2 e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 + \cdots,$$
(25)

Therefore, the asymptotic error constant given by (25) is expressed as

$$AEC(0) = c_2 e_n^2 + \cdots.$$
(26)

It is well known that (26) is the asymptotic error constant for the classical Newton method. Therefore, we take error equation (26) as our next term of the generating series given in (5). Furthermore, we obtain a family of higher iterative method by increasing the terms of summation series of (6). Hence, we show the asymptotic error constant $AEC(k)$ for the $(k+2)$-order Newton-type method. In order to obtain the following terms of the generating series, we must substitute the essential parts in $AEC(k)$, thus

$$c_2 \to t_1, \quad c_3 \to t_2, \quad c_4 \to t_3, \quad c_5 \to t_4, \quad c_6 \to t_5, \quad \cdots \tag{27}$$

$$e_n \to r. \tag{28}$$

where r and t_i are given by (7). We use this principle to obtain the following one-point $(k+2)$-order iterative method;

Second-order Polynomial equation

This method has been recently presented in [8] and we review the basic expansion,

1. $k=1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + \sum_{i=1}^{k} AEC(i-1) r^i\right] = x_n - r\left[1 + t_1 r\right] \tag{29}$$

and the error equation

$$AEC(1) = 2c_2^2 e_n^3 + \cdots \tag{30}$$

2. $k=2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + 2t_1^2 r^2\right] \tag{31}$$

and the error equation

$$AEC(2) = 5c_2^3 e_n^4 + \cdots. \tag{32}$$

3. $k=3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + 2t_1^2 r^2 + 5t_1^3 r^3\right] \tag{33}$$

and the error equation

$$AEC(3) = 14c_2^4 e_n^5 + \cdots. \tag{34}$$

4. $k=4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + 2t_1^2 r^2 + 5t_1^3 r^3 + 14t_1^4 r^4\right] \tag{35}$$

and the error equation

$$AEC(4) = 42c_2^5 e_n^6 + \cdots. \tag{36}$$

5. $k=5$: One-point seventh-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + 2t_1^2 r^2 + 5t_1^3 r^3 + 14t_1^4 r^4 + 42t_1^5 r^5\right] \quad (37)$$

and the error equation

$$AEC(5) = 132c_2^6 e_n^7 + \cdots. \quad (38)$$

6. $k = 6$: One-point eighth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + 2t_1^2 r^2 + 5t_1^3 r^3 + 14t_1^4 r^4 + 42t_1^5 r^5 + 132t_1^6 r^6\right] \quad (39)$$

and the error equation

$$AEC(6) = 429c_2^7 e_n^8 + \cdots. \quad (40)$$

It is well established that the maximum order of convergence of optimal methods with three functions evaluations is four. As illustrated above we have obtained order of convergence greater than four, hence the Kung and Traub conjecture fails for $k \geq 3$. To produce the next one-point higher order of convergence with only three function evaluations, we use the following principle. The process is very simple, using the coefficient of the error equation $AEC(k)$ as our coefficient of the next term of the generating series, thus we can calculate the next higher order of convergence method. Hence, $AEC(k)$ is the error equation of the $(k+2)$-order Newton-type method.

Third-order Polynomial equation

This cubic equation method has been recently presented in [9] and review some of the results of the method.

1. $k = 1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r[1 + t_1 r] \quad (41)$$

and the error equation

$$AEC(1) = \left(2c_2^2 - c_3\right)e_n^3 + \cdots \quad (42)$$

2. $k = 2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2\right] \quad (43)$$

and the error equation

$$AEC(2) = 5c_2\left(c_2^2 - c_3\right)e_n^4 + \cdots \quad (44)$$

3. $k = 3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + 5t_1\left(t_1^2 - t_2\right)r^3\right] \quad (45)$$

and the error equation

$$AEC(3) = \left(14c_2^4 + 3c_3^2 - 21c_2^2 c_3\right)e_n^5 + \cdots \quad (46)$$

4. $k = 4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + 5t_1\left(t_1^2 - t_2\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2 t_2\right)r^4\right] \quad (47)$$

and the error equation

$$AEC(4) = 14c_2\left(3c_2^4 + 2c_3^2 - 6c_2^2 c_3\right)e_n^6 + \cdots \quad (48)$$

5. $k = 5$: One-point seventh-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + 5t_1\left(t_1^2 - t_2\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2 t_2\right)r^4 + 14t_1\left(3t_1^4 + 2t_2^2 - 6t_1^2 t_2\right)r^5\right] \quad (49)$$

and the error equation

$$AEC(5) = 6\left(22c_2^6 - 2c_3^3 + 30c_2^2 c_3^2 - 55c_2^4 c_3\right)e_n^7 + \cdots \quad (50)$$

6. $k = 6$: One-point eighth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + 5t_1\left(t_1^2 - t_2\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2 t_2\right)r^4 \right.$$
$$\left. + 14t_1\left(3t_1^4 + 2t_2^2 - 6t_1^2 t_2\right)r^5 + 6\left(22t_1^6 - 2t_2^3 + 30t_1^2 t_2^2 - 55t_1^4 t_2\right)r^6\right] \quad (51)$$

and the error equation

$$AEC(6) = 3c_2\left(143c_2^6 - 55c_3^3 + 330c_2^2 c_3^2 - 429c_2^4 c_3\right)e_n^8 + \cdots \quad (52)$$

Due to Kung and Traub conjecture the maximum order of convergence of optimal methods with four functions evaluations is eight. As illustrated above when $k \geq 7$ we have obtained order of convergence greater than eight, hence the Kung and Traub conjecture fails.

Fourth-order Polynomial equation

1. $k = 1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r\right] \quad (53)$$

and the error equation

$$AEC(1) = \left(2c_2^2 - c_3\right)e_n^3 + \cdots \quad (54)$$

2. $k = 2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2\right] \quad (55)$$

and the error equation

$$AEC(2) = \left(5c_2^3 - 5c_2 c_3 + c_4\right)e_n^4 + \cdots \quad (56)$$

3. $k = 3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1 t_2 + t_3\right)r^3\right] \quad (57)$$

and the error equation

$$AEC(3) = \left(14c_2^4 + 3c_3^2 - 21c_2^2 c_3 + 6c_2 c_4\right)e_n^5 + \cdots \quad (58)$$

4. $k = 4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3\right)r^4\right]$$

(59)

and the error equation

$$AEC(4) = \left(42c_2^5 + 28c_2c_3^2 - 84c_2^3c_3 - 7c_3c_4 + 28c_2^2c_4\right)e_n^6 + \cdots$$

(60)

5. $k = 5$: One-point seventh-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3\right)r^4 + \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3\right)r^5\right]$$

(61)

and the error equation

$$AEC(5) = \left(132c_2^6 - 12c_3^3 + 180c_2^2c_3^2 - 330c_2^4c_3 + 4c_4^2 + 120c_2^3c_4 - 72c_2c_3c_4\right)e_n^7 + \cdots$$

(62)

6. $k = 6$: One-point eighth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3\right)r^4 + \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3\right)r^5 + \left(132t_1^6 - 12t_2^3 + 180t_1^2t_2^2 - 330t_1^4t_2 + 4t_3^2 + 120t_1^3t_3 - 72t_1t_2t_3\right)r^6\right]$$

(63)

and the error equation

$$AEC(6) = \left(429c_2^7 - 165c_2c_3^3 + 990c_2^3c_3^2 - 1287c_2^5c_3 + 495c_2^4c_4 + 45c_2c_4^2 + 45c_3^2c_4 - 495c_2^2c_3c_4\right)e_n^8 + \cdots$$

(64)

The maximum order of convergence of optimal methods with five functions evaluations is sixteen. To establish the order of convergence greater than sixteen we have to display next fifteen tedious terms, hence we have omitted them and it is clear that the Kung and Traub conjecture fails when $k \geq 15$.

Fifth-order Polynomial equation

1. $k = 1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r\right]$$

(65)

and the error equation

$$AEC(1) = \left(2c_2^2 - c_3\right)e_n^3 + \cdots$$

(66)

2. $k = 2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2\right]$$

(67)

and the error equation

$$AEC(2) = \left(5c_2^3 - 5c_2c_3 + c_4\right)e_n^4 + \cdots$$

(68)

3. $k = 3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3\right] \quad (69)$$

and the error equation

$$AEC(3) = \left(14c_2^4 + 3c_3^2 - 21c_2^2c_3 + 6c_2c_4 - c_5\right)e_n^5 + \cdots \quad (70)$$

4. $k = 4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4\right] \quad (71)$$

and the error equation

$$AEC(4) = \left(42c_2^5 + 28c_2c_3^2 - 84c_2^3c_3 - 7c_3c_4 + 28c_2^2c_4 - 7c_2c_5\right)e_n^6 + \cdots \quad (72)$$

5. $k = 5$: One-point seventh-order iterative method is given by

$$\begin{aligned} x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 \\ &+ \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 + \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4\right)r^5\Big] \end{aligned} \quad (73)$$

and the error equation

$$AEC(5) = \left(132c_2^6 - 12c_3^3 + 180c_2^2c_3^2 - 330c_2^4c_3 + 4c_4^2 + 120c_2^3c_4 - 72c_2c_3c_4 + 8c_3c_5 - 36c_2^2c_5\right)e_n^7 + \cdots \quad (74)$$

6. $k = 6$: One-point eighth-order iterative method is given by

$$\begin{aligned} x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 \\ &+ \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4\right)r^5 \\ &+ \left(132t_1^6 - 12t_2^3 + 180t_1^2t_2^2 - 330t_1^4t_2 + 4t_3^2 + 120t_1^3t_3 - 72t_1t_2t_3 + 8t_2t_4 - 36t_1^2t_4\right)r^6\Big] \end{aligned} \quad (75)$$

and the error equation

$$\begin{aligned} AEC(6) = \Big(&429c_2^7 - 165c_2c_3^3 + 990c_2^3c_3^2 - 1287c_2^5c_3 + 495c_2^4c_4 + 45c_2c_4^2 \\ &+ 45c_3^2c_4 - 495c_2^2c_3c_4 - 9c_4c_5 - 165c_2^3c_5 + 90c_2c_3c_5\Big)e_n^8 + \cdots \end{aligned} \quad (76)$$

The maximum order of convergence of optimal methods with six functions evaluations is thirty-two. To establish the order of convergence greater than thirty-two we have to display next thirty-one tedious terms, hence we have omitted them and it is clear that the Kung and Traub conjecture fails when $k \geq 31$.

Sixth-order Polynomial equation

1. $k = 1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r\right] \quad (77)$$

and the error equation

$$AEC(1) = \left(2c_2^2 - c_3\right)e_n^3 + \cdots \quad (78)$$

2. $k = 2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2\right]$$
(79)

and the error equation

$$AEC(2) = \left(5c_2^3 - 5c_2c_3 + c_4\right)e_n^4 + \cdots$$
(80)

3. $k = 3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3\right]$$
(81)

and the error equation

$$AEC(3) = \left(14c_2^4 + 3c_3^2 - 21c_2^2c_3 + 6c_2c_4 - c_5\right)e_n^5 + \cdots$$
(82)

4. $k = 4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4\right]$$
(83)

and the error equation

$$AEC(4) = \left(42c_2^5 + 28c_2c_3^2 - 84c_2^3c_3 - 7c_3c_4 + 28c_2^2c_4 - 7c_2c_5 - 5c_6\right)e_n^6 + \cdots$$
(84)

5. $k = 5$: One-point seventh-order iterative method is given by

$$\begin{aligned}x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 \\ &+ \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 + \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4 - 5t_5\right)r^5\Big]\end{aligned}$$
(85)

and the error equation

$$AEC(5) = \left(132c_2^6 - 12c_3^3 + 180c_2^2c_3^2 - 330c_2^4c_3 + 4c_4^2 + 120c_2^3c_4 - 72c_2c_3c_4 + 8c_3c_5 - 36c_2^2c_5 - 40c_2c_6\right)e_n^7 + \cdots$$

(86)

6. $k = 6$: One-point eighth-order iterative method is given by

$$\begin{aligned}x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 \\ &+ \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4 - 5t_5\right)r^5 \\ &+ \left(132t_1^6 - 12t_2^3 + 180t_1^2t_2^2 - 330t_1^4t_2 + 4t_3^2 + 120t_1^3t_3 - 72t_1t_2t_3 + 8t_2t_4 - 36t_1^2t_4 - 40t_1t_5\right)r^6\Big]\end{aligned}$$
(87)

and the error equation

$$\begin{aligned}AEC(6) = \Big(&429c_2^7 - 165c_2c_3^3 + 990c_2^3c_3^2 - 1287c_2^5c_3 + 495c_2^4c_4 + 45c_2c_4^2 \\ &+ 45c_3^2c_4 - 495c_2^2c_3c_4 - 9c_4c_5 - 165c_2^3c_5 + 90c_2c_3c_5 + 45c_3c_6 - 225c_2^2c_6\Big)e_n^8 + \cdots\end{aligned}$$
(88)

The maximum order of convergence of optimal methods with seven functions evaluations is sixty-four. To establish the order of convergence greater than sixty-four we have to display next sixty-three tedious terms, hence we have omitted them and it is clear that the Kung and Traub conjecture fails when $k \geq 63$.

Seventh-order Polynomial equation

1. $k = 1$: One-point third-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r\right] \tag{89}$$

and the error equation

$$AEC(1) = \left(2c_2^2 - c_3\right)e_n^3 + \cdots \tag{90}$$

2. $k = 2$: One-point fourth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2\right] \tag{91}$$

and the error equation

$$AEC(2) = \left(5c_2^3 - 5c_2c_3 + c_4\right)e_n^4 + \cdots \tag{92}$$

3. $k = 3$: One-point fifth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3\right] \tag{93}$$

and the error equation

$$AEC(3) = \left(14c_2^4 + 3c_3^2 - 21c_2^2c_3 + 6c_2c_4 - c_5\right)e_n^5 + \cdots \tag{94}$$

4. $k = 4$: One-point sixth-order iterative method is given by

$$x_{n+1} = x_n - r\left[1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4\right] \tag{95}$$

and the error equation

$$AEC(4) = \left(42c_2^5 + 28c_2c_3^2 - 84c_2^3c_3 - 7c_3c_4 + 28c_2^2c_4 - 7c_2c_5 - 5c_6\right)e_n^6 + \cdots \tag{96}$$

5. $k = 5$: One-point seventh-order iterative method is given by

$$\begin{aligned} x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 \\ &+ \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 + \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4 - 5t_5\right)r^5\Big] \end{aligned} \tag{97}$$

and the error equation

$$AEC(5) = \left(132c_2^6 - 12c_3^3 + 180c_2^2c_3^2 - 330c_2^4c_3 + 4c_4^2 + 120c_2^3c_4 - 72c_2c_3c_4 + 8c_3c_5 - 36c_2^2c_5 - 40c_2c_6\right)e_n^7 + \cdots \tag{98}$$

6. $k = 6$: One-point eighth-order iterative method is given by

$$\begin{aligned} x_{n+1} = x_n - r\Big[&1 + t_1 r + \left(2t_1^2 - t_2\right)r^2 + \left(5t_1^3 - 5t_1t_2 + t_3\right)r^3 + \left(14t_1^4 + 3t_2^2 - 21t_1^2t_2 + 6t_1t_3 - t_4\right)r^4 \\ &+ \left(42t_1^5 + 28t_1t_2^2 - 84t_1^3t_2 - 7t_2t_3 + 28t_1^2t_3 - 7t_1t_4 - 5t_5\right)r^5 \\ &+ \left(132t_1^6 - 12t_2^3 + 180t_1^2t_2^2 - 330t_1^4t_2 + 4t_3^2 + 120t_1^3t_3 - 72t_1t_2t_3 + 8t_2t_4 - 36t_1^2t_4 - 40t_1t_5\right)r^6\Big] \end{aligned} \tag{99}$$

and the error equation

$$\begin{aligned} AEC(6) = \Big(&429c_2^7 - 165c_2c_3^3 + 990c_2^3c_3^2 - 1287c_2^5c_3 + 495c_2^4c_4 + 45c_2c_4^2 \\ &+ 45c_3^2c_4 - 495c_2^2c_3c_4 - 9c_4c_5 - 165c_2^3c_5 + 90c_2c_3c_5 + 45c_3c_6 - 225c_2^2c_6 + 180c_2c_7\Big)e_n^8 + \cdots \end{aligned} \tag{100}$$

The maximum order of convergence of optimal methods with eight functions evaluations is 128. To establish the order of convergence greater than 128 we have to display next 127 tedious term, hence we have omitted them and it is clear that the Kung and Traub conjecture fails when $k \geq 127$. The above procedure may be repeated for higher-order of polynomial equation.

Remark 1

The new one-point iterative method requires *n* function evaluations and has the order of convergence $(k+2)$. To determine the efficiency index of the new method, definition 2 shall be used. Hence, the efficiency index of the new iterative method given by (11) is $\sqrt[n]{k+2}$ and the efficiency index of the classical Newton method is $\sqrt[2]{2}$. For particular set of values of *k*, we have shown that the efficiency index of the new one-point method is much better than the other similar methods which are based on the Kung and Traub conjecture.

Remark 2

Kung and Traub conjecture fails when the total number of terms τ required in the generating series (6) is given by

$$\tau_d = 2^d - 1 \tag{101}$$

where *d* is the degree of the polynomial equation.

CONCLUSION

In this work, a new one-point $(k+2)$-order Newton-type method has been presented. The prime motive for presenting the new class of iterative method was to extend and demonstrate the use of the Thukral's quadratic and cubic equation methods recently introduced in [8,9]. Furthermore, demonstrate that the Kung and Traub conjecture fails. It is very simple to evaluate the terms of the generating series (6), hence we can construct any desire order of convergence. Recently we have established the essential advantages of the new one-point method are: very high computational efficiency; the new method is not limited to the Kung and Traub conjecture; better efficiency index than the classical Newton method; simple one-step iteration, not limited to quadratic and cubic equations [1,2,8,9], simple to construct, efficient and robust. Finally, further investigation is needed to improve and reduce the number of terms required for the Kung and Traub conjecture to fail.

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