AHMES' METHOD TO SQUARING THE CIRCLE

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ABSTRACT

Squaring the circle is the problem which accompanies human civilization from its beginning. Two co-existed cultures Babylonian (Sumerian) and Egyptian developed radically different approaches to calculate the area of the circle. Sumerians used length of the circumference to determine the area of the circle. Ancient Egyptians used the diameter of the circle to find its area. In ancient Egypt, in 1650 B.C., scribe Ahmes wrote on papyrus: "Cut off 1/9 of a diameter and construct a square upon the remainder; this has the same area as the circle." The geometrical realization of this recipe needs an additional mathematical knowledge: How to divide a given segment into 9 equal pieces. Is it possible that other divisions of a diameter than 8/9 are much better? Ahmes' method to calculate the area of the circle is a powerful pedagogical tool connecting mathematics with historical contexts.

Keywords: Ahmes, civilization, Egypt, number pi, squaring the circle.

INTRODUCTION

The general problem of constructing a square with the same area as a given circle is called the quadrature of the circle or squaring the circle. It has been known since the year 1822 (Lindemann 1822) that the quadrature of a circle with straightedge (without marked units on it) and compass is impossible to accomplish. Of course, always for any circle exists a square of the same area. It is easy to determine this square. For a given circle of area A, its corresponding square has the side equals to the square root of A. The quadrature of a circle is possible but not under restriction defined by the ancient Greek geometers (Euclid). Only allowed tools to use are a straightedge and compass, and it is only allowed to do this task in finite number of steps.

Another approach to realize the quadrature is to perform its approximation. In this situation the constructed square has almost the same area as a given circle. One such method appears in one of the earliest mathematical document known as the Rhind papyrus or Ahmes papyrus (Spalinger 1990, Clagett 1999). The papyrus was found in Thebes (now Luxor, Egypt) in the ruins of a small building near the Ramesseum. The papyrus was written by an Egyptian scribe A'h-mosè, now in modern time called Ahmes. The Ahmes papyrus has been dated to about 1650 B.C. There is only one older mathematical papyrus, called the Moscow papyrus, dated two hundred years earlier on 1850 B.C. The Ahmes papyrus was purchased in 1858 in Egypt in the Luxor market by Alexander Henry Rhind, a Scottish lawyer, antiquarian and egyptologist, and acquired by the British Museum after his death. Suffering from pulmonary disease, Rhind travelled to Egypt. He was fascinated by culture of ancient Egypt.

It appears that Ahmes wrote a copy of an older document. The scroll consists of 87 problems. Ahmes presents many problems with solutions and gives the examples of practical applications (Gillings 1972, Robins and Shute 1987). The problems and their solutions are phrased in terms of specific numbers. The papyrus is primary source of ancient Egyptian mathematics. The scribe Ahmes wrote the following in the beginning of the document in its presentation: "*Accurate reckoning of entering into things, knowledge of existing things all, mysteries, secrets all*." He continues with: "*This book was copied in regnal year 33, month 4 of Akhet, under the majesty of the King of Upper and Lower Egypt, Awserre, given life, from an ancient copy made in the time of the King of Upper and Lower Egypt Nimaatre. The scribe Ahmose writes this copy*."

The papyrus is written in hieratic script, a cursive writing system, and it is a single roll which was originally about 5.4 meters long by 32 centimeters wide (approximately 18 feet by 13 inches).

THE AHMES METHOD TO CALCULATE AREA

There are a few problems in the papyrus related to circle and its area. Our concern is with problem 50 in the Ahmes papyrus which reads:

A circular field has diameter 9 khet. What is its area?

A khet is a length measurement of about 50 meters. Thus the circle has diameter of 450 meters. In our time we can easily calculate its area using the number π and the well-known formula ($A = \pi r^2$) (.

Ahmes' solution is as follows:

Take away thou 1/9 of it, namely 1; the remainder is 8. Make thou the multiplication 8 times 8; becomes it 64; the amount of it, this is, in area 64 setat.

We may take this solution as a general formula:

"*Cut off 1/9 of a diameter and construct a square upon the remainder; this has the same area as the circle.*"

Then according to this recipe, in modern notation, we obtain the "formula" for the area A of a circle of diameter d, where d=2r, and r is a radius of the circle:

$$A = \pi r^2 \approx \left(\frac{8}{9}d\right)^2 = \left(\frac{8}{9}2r\right)^2 = \left(\frac{16}{9}\right)^2 r^2.$$

From this equation, after eliminating the term with r to power two, we have the following approximate value for the number π.

$$\pi \approx \left(\frac{16}{9}\right)^2 = \left(\frac{4}{3}\right)^4 = \frac{256}{81} = 3.16049382716049 \dots .$$

No so bad approximation in such old time (Engels 1977, Gerdes 1985). This method can be realized geometrically. Actually it is very good practical method to squaring the circle. Imagine that we have a given circle drawn on the ground. Ahmes' method allows easily to draw the corresponding square. Figure 1 shows the situation described in the problem 50 and its solution.

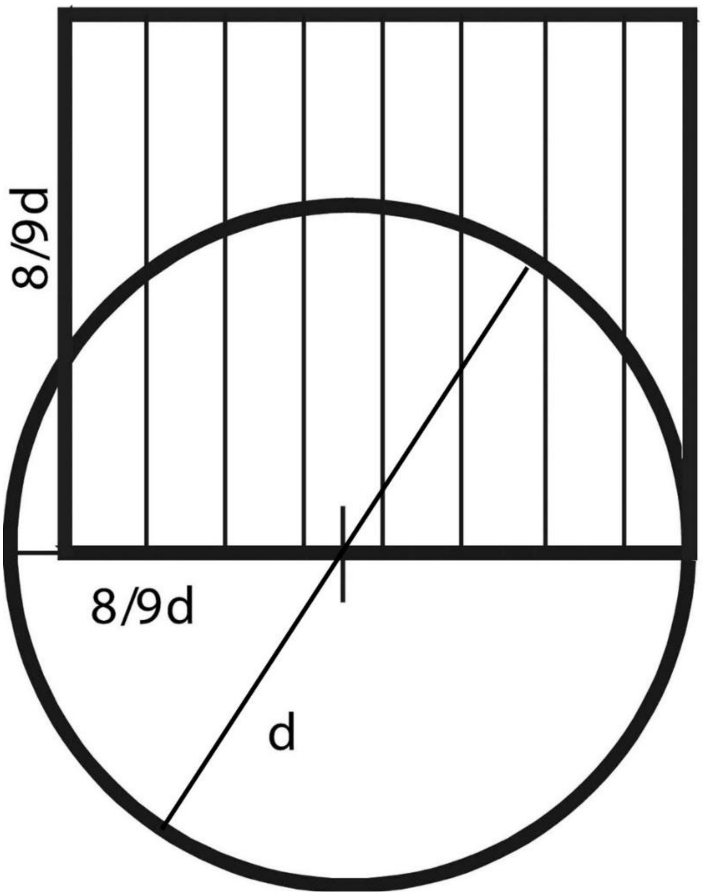

Figure 1. The method used in ancient Egypt to squaring a given circle.

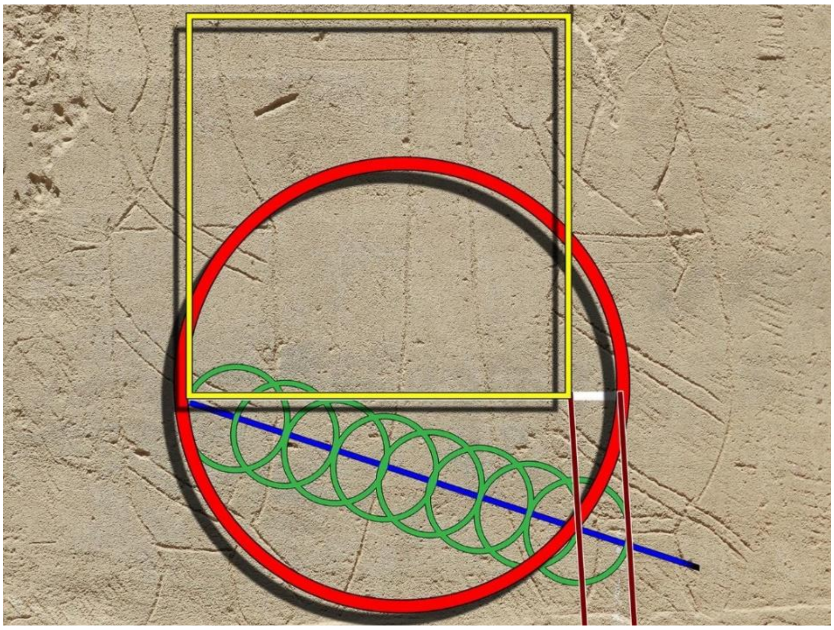

Figure 2. The realization of the Ahmes method. We need to divide the diameter d into 9 parts.

A geometrical construction of the square according to Ahmes has one important difficulty. We have to be able to divide the diameter into 9 equal parts. It is the geometrical construction in itself. For this purpose we may apply the intercept theorem. This theorem is known in elementary geometry as Thales' theorem. It is about the ratios of various line segments that are created if two intersecting lines are intercepted by a pair of parallels (Euclid; The

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proposition 13 of the book III indicates that: "*Any parallel DE to one of the sides BC of a triangle ABC, divides the other sides AB and AC, in proportional parts*"). Here we used it to divide the diameter into 9 equal parts (See Figure 2).

OTHER DIVISIONS OF THE DIAMETER

By a consequence, the following question occurs. Is it possible that other divisions of the diameter are geometrically easier and give better approximation of the number π? Let's investigate it. Is it possible that the fraction x is better than 8/9? Gives better approximation to the number π and/or is much easier to realize its geometrical construction. We have the following condition and equation on the value of x.

$$A = \pi r^2 = (xd)^2 = (x2r)^2 = 4x^2r^2.$$

$$x = \sqrt{\frac{\pi}{4}} = 0.886226925452758\cdots .$$

Now we have to find convenient approximation to the value of x. For this purpose we are using a continued fraction technique (Rocket and Szüsz 1992). The simple continued fraction for x generates all of the best rational approximations for x. To determine the corresponding continued fraction we used the R software. It is freely available programing language (R Core Team 2016). The listing of the short program to realize this task is here presented. We obtained the following approximation by continued fraction [0;1, 7, 1, 3, 1, 2, 1, 57,…] using only nine terms. Table 1 shows the values of the best approximations by simple fractions derived from the obtained continued fraction.

Table 1. The best rational approximations to the value of x and the corresponding estimated π.

Numerator	**Denominator**	**Approximation to x**	**Approximation to π**
0	1	0/1=0	0
1	1	1/1=1	4
7	8	7/8=0.8750	3.0625
8	9	8/9=0.888888888888889	3.160493827160494
31	35	31/35=0.885714285714286	3.137959183673469
39	44	39/44=0.886363636363636	3.142561983471075
109	123	109/123=0.886178861788618	3.141251900323882
148	167	148/167=0.886227544910180	3.141597045430100
8545	9642	8545/9642=0.886226923874715	3.141592642401758

The results show the following interesting facts: (i). the value 8/9 is present among the best rational approximations, (ii). the value 7/8 is easy to realize geometrically but only gives π=3.0625, (iii). other fractions are difficult to implement geometrically as their denominators are large numbers, (iv). using the fraction 8545/9642 we have π=3.14159264… .

It is very intriguing that the value 8/9 occurred among the best rational approximations. We don't know exactly how the Ahmes method was elaborated but scribe Ahmes was right to use 8/9 (Engels 1977, Gerdes 1985). Next the best value is 31/35, but gives 3.137… . Another interesting part, left to a reader, is to find the best approximation of the value x by the fraction a/b, where b=2, 4, 8, 16,… . We have one such type in the determined series, i.e. 7/8. These fractions (with their denominator as a power of 2) are more convenient in geometrical construction of the Ahmes method, as they need only a series of halving of the diameter.

PROGRAM IN R

Listing of the program in R. The program calculates continued fractions for x.

```
#Program EgyptPi #Author: M. Szyszkowicz
library(contfrac); options(digits=15)
x=sqrt(pi/4); print(x); # x= 0.886226925452758
fracx = as_cf(x, n = 9)
print(fracx)  #continued fraction
# [0  1  7  1  3  1  2  1 57]
fraction = convergents(fracx)
print(fraction$A) #Numerators  :  0    1    7    8    31    39   109   148 8545
print(fraction$B) #Denominators:1    1    8    9    35    44   123   167 9642
for (k in 1:9){
N=fraction$A[k]; D=fraction$B[k]
w=N/D*2; aprPi=w*w
result =c(N/D,aprPi); print(result); }; # The end
```

CONCLUSIONS

We do not know how the Ahmes method was derived. There are many guesses and speculations on this subject. As we see the geometrical realization of this method needs an additional geometrical knowledge – to know how to divide a segment into 9 equal parts. As we see among the presented fractions, 8/9 was one of the best choice.

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