THE PULLBACK ATTRACTORS FOR THE HIGHER-ORDER KIRCHHOFF-TYPE EQUATION WITH NONLINEAR STRONGLY DAMPED TERM AND DELAYS*

Guoguang Lin & Yuting Sun+
Department of Mathematics, Yunnan University
Kunming, Yunnan 650091
PEOPLE'S REPUBLIC OF CHINA

ABSTRACT

We investigate the pullback attractors for the Higher-order Kirchhoff-type equation with nonlinear strongly damping and delays:
$\frac{\partial^2 u}{\partial t^2}+\sigma(\|\nabla^m u^2\|)(-\Delta)^m\frac{\partial u}{\partial t}+\phi(\|\nabla^m u\|^2)(-\Delta)^m u=f(x)+h(t,u_t)$. For strong nonlinear damping σ and ϕ, we make assumptions (A_1)-(A_2). For delay forcing term h, we make assumptions (G_1)-(G_2). Under of the proper assume, the main results are existence and uniqueness of the solution are proved by Galerkin method, and deal with the pullback attractors.

Keywords: strongly nonlinear damped, Higher-order Kirchhoff equation, the pullback attractors, delays.

2010 Mathematics Classification: 35B41, 35G31

1. INTRODUCTION

We consider the following Higher-order Kirchhoff-type equation:

$$\frac{\partial^2 u}{\partial t^2}+\sigma(\|\nabla^m u\|^2)(-\Delta)^m\frac{\partial u}{\partial t}+\phi(\|\nabla^m u\|^2)(-\Delta)^m u=f(x)+h(t,u_t), t>\tau, \quad (1.1)$$

$$u|_{\partial\Omega}=0, t\geq\tau-r, \quad (1.2)$$

$$u(x,t)=\psi(x,t-\tau), \frac{\partial u}{\partial t}(x,t)=\frac{\partial\psi}{\partial t}(x,t-\tau), x\in\Omega, t\in[\tau-r,\tau] \quad (1.3)$$

where $m>1$ is an integer constant, and Ω is a bounded domain of R^n with a smooth dirichlet boundary $\partial\Omega$ and initial value. σ and ϕ are

*This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 11561076.
+Email: gglin@ynu.edu.cn. syt19911006@163.com.

scalar functions specified later, $f(x)+h(t,u_t)$ is the source intensity which may depend on the history of the solution, ψ is the initial value on the interval $[\tau-r,\tau]$ where $r>0$. Moreover, u_t is defined for $\theta\in[-r,0]$ as $u_t(\theta)=u(t+\theta)$.

This kind of wave models goes back to G. Kirchhoff [1] and has been studied by many authors under different types of hypotheses. It's well known that the long time behavior of many dynamical system generated by evolution equations can be described naturally in term of attractors of corresponding semigroups. Attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation. There have been many researches on the long-time behavior of solutions to the nonlinear damped wave equations with delays. The existence of global and exponential attractors have been investigated by many authors[see 2-5]. A new type of attractor, called a pullback attractor, was proposed and investigated for non-autonomous or these random dynamical systems. The pullback attractor describing this attractors to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value[see 6-14].

In [6], the existence of a pullback attractor is proved for a damped wave equation containing a delay forcing term.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}-\Delta u=f+h(t,u_t),t>\tau,\\
&u|_\Gamma=0,t\ge\tau-r,\\
&u(x,t)=\phi(x,t-\tau),x\in\Omega,t\in[\tau-r,\tau],\\
&\frac{\partial u}{\partial t}=\frac{\partial\phi}{\partial t},x\in\Omega,t\in[\tau-r,\tau].
\end{aligned}\tag{1.4}$$

The result follows from the existence of a compact set which is uniformly attracting for the two-parameter semigroup associated to the model.

In 2013, Guoguang Lin, Fangfang Xia and Guigui Xu [7] study the global and pullback attractors for a strongly damped wave equation with delays when the force term belongs to different space.

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}-\beta\Delta\frac{\partial u}{\partial t}-\Delta u+g(u)=f(x)+h(t,u_t),t>\tau.\tag{1.5}$$

The results following from the solution generate a compact set.

In 2010, Maria Anguiano [8] studied the existence of a pullback attractor in $L^2(\Omega)$ for the following non-autonomous reaction-diffusion equation

$$\begin{aligned}&\frac{\partial u}{\partial t}-\Delta u=f(u)+h(t), \quad in \quad \Omega\times(\tau,+\infty),\\&u=0 \quad on \quad \partial\Omega\times(\tau,+\infty),\\&u(x,\tau)=u_\tau(x), x\in\Omega.\end{aligned} \tag{1.6}$$

is proved in this paper, when the domain Ω is not necessarily bounded but satisfying the Poincaré inequality, and $h\in L^2_{loc}(R;H^{-1}(\Omega))$. The main concept used in the proof is the asymptotic compactness of the process generated by the problem.

In 2006, Yejuan Wang [9] present the necessary and sufficient conditions and a new method to study the existence of pullback attractors of non-autonomous infinite dimensional dynamical systems. For illustrating the method, they apply it to non-autonomous 2D Navier-Stokes systems.

$$\begin{aligned}&u'+a(t)\sum_{i=1}^{2}u_i\partial_i u=\nu\Delta u-\nabla p+f(t).\\&divu=0,\\&u|_{\partial\Omega}=0,\end{aligned} \tag{1.7}$$

where $\nu>0$ and Ω is a smooth bounded domain in R^2.

They also show that the para-metrically inflated pullback attractors and uniform attractors are robust with respect to the perturbations of both cocycle mappings and diving systems. They take an example, and consider the non-autonomous 2D Navier-Stokes system with rapidly oscillating external force.

For equations of the form

$$\begin{aligned}&u'+A(t)u(t)=F(t,u_t),t\ge 0.\\&u(t)=\psi(t),t\in[-h,0].\end{aligned} \tag{1.8}$$

have been analized by some people. For example, M.J.Garrido and J.Real [10] prove some results on the existence and uniqueness of solution for a class of evolution equations of second order in time, containing some hereditary characteristics. Their theory is developed from a variational point of view, and in general functional setting which permits us to deal with several kinds of delay terms. In particular, we can consider terms which contain spatial partial derivatives with deviating arguments.

At present, most Higher-order Kirchhoff-type equations investigate global attractors, exponential attractors and blow-up of solution. However, we investigate the pullback attractors of the Higher-order Kirchhoff-type equation (1.1) with strong nonlinear damping and delays. In section 2, we introduce some basic concepts. Under of the proper assume, in section 3, we prove the existence and uniqueness of the solution by Galerkin method. At last, in section 4, we obtain the existence of the pullback attractor. The technology we use is introduced in [6], that is, we divide the semigroup into two: the one is a asymptotically close to 0, while the other is uniformly compact, so we can get the pullback attractor.

2. PRELIMINARIES

Denote and assumptions

For brevity, we denote the simple symbol, $H = L^2(\Omega), V = H_0^m(\Omega)$,

$D(A) = H_0^{m+1}(\Omega)$, $f = f(x)$ and $h = h(t, u_t)$. $\|\cdot\|$ represents inner product, $c_i (i = 0,1,\ldots 5)$ are constants, $\mu_i (i = 0,1)$ are also constants. λ is the first eigenvalue of the operator $-\Delta$.

Our problem can be written as a second order differential equation in H:

$$u'' + \sigma(\|\nabla^m u\|^2)(-\Delta)^m u' + \phi(\|\nabla^m u\|^2)(-\Delta)^m u = f(x) + h(t, u_t), t > 0, \quad (2.1)$$

$$u(t) = \psi(t-\tau), t \in [\tau - r, \tau], \quad (2.2)$$

$$u'(t) = \psi(t-\tau), t \in [\tau - r, \tau]. \quad (2.3)$$

In general, if $(X, \|\cdot\|_X)$ is a Banach space, we denote by C_X the space $C^0([-r,0]; X)$ with the sup-norm, i.e. $\|\psi\|_{C_X} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|_X$, for $\psi \in C_X$. Given another Banach space $(Y, \|\cdot\|_Y)$ such that the injection $X \subset Y$ is continuous, we denote by $C_{X,Y}$ the Banach space $C_X \cap C^1([-r,0]; Y)$ with the norm $\|\cdot\|_{C_{X,Y}}$ defined by

$$\|\psi\|_{C_{X,Y}}^2 = \|\psi\|_{C_X}^2 + \|\psi_t\|_{C_Y}^2, \ for \quad \psi \in C_{X,Y}. \quad (2.4)$$

We will use the space C_X, C_Y, $C_{V,H}$ and $C_{D(A),H_0^1}$ in our analysis.

In this section, we present some assumptions needed in the proof of our results. For this reason, we assume that

(A$_1$) setting $\phi'(s) - \varepsilon\sigma'(s) > 0, \sigma(s) > 1, \forall \alpha > 0$, then

$$\begin{aligned}
&(\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^m u\|^2 \\
&\geq \int_0^{\|\nabla^m u\|^2} \phi(s) - \varepsilon\sigma(s)\, \mathrm{d}s \quad (2.5) \\
&\geq \alpha\|\nabla^m u\|^2 + c_0.
\end{aligned}$$

(A2) $\mu_0 < \phi(s) < \mu_1,\ \mu = \begin{cases} \mu_0, \dfrac{d}{dt}\|\nabla^m w\|^2 \geq 0 \\ \mu_1, \dfrac{d}{dt}\|\nabla^m w\|^2 < 0. \end{cases}$ (2.6)

We make the following hypotheses on the function h: $R \times C_H \to H$.

(G1) $\forall \xi \in C_H, t \in R \to h(t,\xi) \in H$ is continuous; (2.7)

(G2) $\forall t \in R,\ h(t,0) = 0;$ (2.8)

(G3) $\exists L_h > 0, such\ that\ \forall t \in R, \forall \xi, \eta \in C_H$

$$\|h(t,\xi) - h(t,\eta)\| \leq L_h \|\xi - \eta\|_{C_H}; \qquad (2.9)$$

(G4) $\exists m_0 > 0,\ C_h > 0$, such that $\forall m \in [0, m_0],\ \tau < t$, and $u, v \in C^0([\tau - r, t]; H)$,

$$\int_\tau^t e^{ms} \|h(s,u_s) - h(s,v_s)\|^2 \,\mathrm{d}\, s \leq C_h^2 \int_{\tau-r}^t e^{ms} \|u(s) - v(s)\|^2 \,\mathrm{d}\, s; \qquad (2.10)$$

(G5) $h \in C^1(R \times C_H; H)$, and there exists $C > 0$ such that, for any $(t,\xi) \in R \times C_H$, the Fréchet derivative $\delta h(t,\xi) \in L(R \times C_H, H)$ satisfies

$$\|\delta h(t,\xi)\|_{L(R\times C_H,H)} \leq C(1 + \|\xi\|_{C_H}); \qquad (2.11)$$

Preliminaries on pullback attractors

In [2], we deal with the global attractors by semigroup $S(t)$. Instead of a family of the one-parameter map $S(t)$, we need to use a two-parameter semigroup or process $U(t,\tau)$ on the complete metric space X, $U(t,\tau)\psi$ denotes the value of the solution at time t which was equal to the initial value ψ at time τ.

The semigroup property is replace by the process composition property

$$U(t,\tau)U(\tau,r)=U(t,r), \qquad for \quad all \quad t\ge\tau\ge r, \qquad (2.12)$$

and obviously, the initial condition implies $U(\tau,\tau)=Id$.

Definition 2.1 [7] Let U be the two-parameter semigroup or process on the complete metric space X. A family of compact set $A(t)_{t\in R}$ is said to be a pullback attractor for U, if, for all $\tau\in R$. It satisfies

(1) $U(t,\tau)A(\tau)=A(\tau)$, for all $t\ge\tau$;

(2) $\lim_{s\to\infty} dist_X(U(t,t-s)D, A(t))=0$, for all bounded $D\subset X$, and all $t\in R$.

Definition 2.2 [7] If the family $B(t)_{t\in R}$ satifying

(1) pullback absorbing with respect to the process U, if for all $t\in R$ and all bounded $D\subset X$, there exists $T_D(t)>0$ such that $U(t,t-s)D\subset B(t)$ for all $s>T_D(t)$;

(2) pullback attracting with respect to the process U, if for all $t\in R$, all bounded $D\subset X$, and all $\varepsilon>0$, there exists $T_{\varepsilon,D}(t)>0$ such that for all $s>T_{\varepsilon,D}(t)$

$$dist_X(U(t,t-s)D, B(t))<\varepsilon; \qquad (2.13)$$

(3) pullback uniformly absorbing (respectively uniformly attracting) if $T_D(t)$ in pact (a) (respectively $T_{\varepsilon,D}(t)$ in part (b)) does not depend on the time t.

Theorem 2.1[7] Let $U(t,\tau)$ be a two-parameter process, and suppose $U(t,\tau): X\to X$ is continuous for all $t\ge\tau$. If there exists a family of compact pullback attracting sets $B(t)_{t\in R}$, then there exists a pullback attractor $A(t)_{t\in R}$, such that $A(t)\subset B(t)$ for all $t\in R$, and which is given by

$$A=\overline{\bigcup_{D\subset X}\Lambda_D(t)}, \quad where \quad \Lambda_D(t)=\bigcap_{n\in N}\overline{\bigcup_{s\ge n}U(t,-s)D}. \qquad (2.14)$$

3 EXISTENCE AND UNIQUENESS OF THE SOLUTION

Theorem 3.1 Assume that $f\in L^2_{loc}(R,H)$, $\psi\in C_{V,H}$, σ and ϕ satisfies (A_1)-(A_2), and h satisfies (G_1)-(G_4). Then, for any $\tau\in R$, there exists a unique solution $u(\cdot)=u(\cdot;\tau,\psi)$ of the problem (1.1) such that $u\in C^0([\tau-t,\infty];V)\bigcap C^1([\tau-r,\infty];H)$.

Proof. Let $v=u'+\varepsilon u$, then equation (2.1) become

$$v' - \varepsilon v + \varepsilon^2 u + \sigma(\|\nabla^m u\|^2)(-\Delta)^m v - \varepsilon\sigma(\|\nabla^m u\|^2)(-\Delta)^m u$$
$$+\phi(\|\nabla^m u\|^2)(-\Delta)^m u = f(x) + h(t,u_t). \quad (3.1)$$

We use v multiply with both sides of equation (3.1) and obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon\|v\|^2 + \varepsilon^2(u,v) + \sigma(\|\nabla^m u\|^2)((-\Delta)^m v, v)$$
$$-\varepsilon\sigma(\|\nabla^m u\|^2)((-\Delta)^m u, v) + \phi(\|\nabla^m u\|^2)((-\Delta)^m u, v) \quad (3.2)$$
$$= (f(x), v) + (h(t,u_t), v)$$

Handle some items of (3.2), as follow

$$\varepsilon^2(u,v) = \varepsilon^2(u, u' + \varepsilon u) = \frac{\varepsilon^2}{2}\frac{d}{dt}\|u\|^2 + \varepsilon^3\|u\|^2 \quad (3.3)$$

$$\sigma(\|\nabla^m u\|^2)((-\Delta)^m v, v) = \sigma(\|\nabla^m u\|^2)\|\nabla^m v\|^2 \quad (3.4)$$

$$(\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))((-\Delta)^m u, v)$$
$$= (\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))((-\Delta)^m u, u' + \varepsilon u)$$
$$= \frac{\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2)}{2}\frac{d}{dt}\|\nabla^m u\|^2 \quad (3.5)$$
$$+\varepsilon(\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^m u\|^2$$

By using Hölder inequality, Young's inequality, we obtain

$$(f(x), v) \le \|f(x)\|\|v\| \le \frac{1}{2\varepsilon}\|f\|^2 + \frac{\varepsilon}{2}\|v\|^2 \quad (3.6)$$

$$(h(t,u_t), v) \le \|h(t,u_t)\|\|v\| \le \frac{1}{2\varepsilon}\|h\|^2 + \frac{\varepsilon}{2}\|v\|^2 \quad (3.7)$$

From the above,we have

$$\frac{1}{2}\frac{d}{dt}[\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s]+\sigma(\|\nabla^m u\|^2)\|\nabla^m v\|^2$$
$$-2\varepsilon\|v\|^2+\varepsilon^3\|u\|^2+\varepsilon(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^m u\|^2 \tag{3.8}$$
$$\le\frac{1}{2\varepsilon}\|f\|^2+\frac{1}{2\varepsilon}\|h\|^2$$

When assumption (A_1) hold, and $\varepsilon<\frac{\lambda^m}{2}$, we receive

$$\frac{1}{2}\frac{d}{dt}[\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s]+(\sigma(\|\nabla^m u\|^2)-1)\|\nabla^m v\|^2$$
$$+(\lambda^m-2\varepsilon)\|v\|^2+\varepsilon^3\|u\|^2+\varepsilon\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s \tag{3.9}$$
$$\le\frac{1}{2\varepsilon}\|f\|^2+\frac{1}{2\varepsilon}\|h\|^2.$$

Due to assumption, we get

$$\frac{d}{dt}[\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s]$$
$$+\gamma_1(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s) \tag{3.10}$$
$$\le\frac{1}{\varepsilon}\|f\|^2+\frac{1}{\varepsilon}\|h\|^2,$$

where $\gamma_1=min\{2\lambda^m-4\varepsilon,2\varepsilon\}$.

Denote $H_1(t)=\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s$.

Because

$$\frac{d}{dt}[e^{mt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s)]$$
$$=me^{mt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s) \tag{3.11}$$
$$+e^{mt}\frac{d}{dt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s).$$

Hence, we can get the following inequality

$$\frac{d}{dt}[e^{mt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s)]$$
$$\le(m-\gamma_1)e^{mt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s) \tag{3.12}$$
$$+\frac{e^{mt}}{\varepsilon}\|f\|^2+\frac{e^{mt}}{\varepsilon}\|h\|^2.$$

By integration over the interval $[\tau,t]$, we deduce

$$\begin{aligned}
& e^{mt}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s)\\
&\le e^{m\tau}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s)\\
&+\int_\tau^t\frac{e^{ms}}{\varepsilon}\|f\|^2\,\mathrm{d}s+\int_\tau^t\frac{e^{ms}}{\varepsilon})\|h(s,u_s\|^2\,\mathrm{d}s\\
&+(m-\gamma_1)\int_\tau^t e^{ms}(\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s)\,\mathrm{d}s\\
&\le e^{m\tau}H_1(t)+(m-\gamma_1)\int_\tau^t e^{ms}H_1(t)\,\mathrm{d}s\\
&+\frac{1}{\varepsilon m}\|f\|^2(e^{mt}-e^{m\tau})+\frac{C_h^2\lambda^{-m}}{\varepsilon}\int_{\tau-r}^t e^{ms}\|\nabla^m u\|^2\,\mathrm{d}s\\
&=e^{m\tau}H_1(t)+(m-\gamma_1)\int_\tau^t e^{ms}H_1(t)\,\mathrm{d}s+\frac{1}{\varepsilon m}\|f\|^2(e^{mt}-e^{m\tau})\\
&+\frac{C_h^2\lambda^{-m}}{\varepsilon}\int_{\tau-r}^\tau e^{ms}\|\nabla^m u\|^2\,\mathrm{d}s+\frac{C_h^2\lambda^{-m}}{\varepsilon}\int_t^\tau e^{ms}\|\nabla^m u\|^2\,\mathrm{d}s
\end{aligned}\tag{3.13}$$

Assumption (A_1) imply

$$\alpha\|\nabla^m u\|<\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s+c_1=H_1(t)+c_1.\tag{3.14}$$

So,we can get the following inequality

$$\|\nabla^m u\|\le\frac{\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}s+c_1}{\alpha}=\frac{H_1(t)+c_1}{\alpha}.\tag{3.15}$$

(3.15) means that

$$\begin{aligned}
&\int_\tau^t e^{ms}\|\nabla^m u(s)\|^2\,\mathrm{d}s\\
&\le\int_\tau^t e^{ms}\frac{H_1(t)}{\alpha}\,\mathrm{d}s+\int_\tau^t e^{ms}\frac{c_1}{\alpha}\,\mathrm{d}s\\
&\le\frac{1}{\alpha}\int_\tau^t e^{ms}H_1(t)\,\mathrm{d}s+\frac{c_1}{\alpha m}(e^{mt}-e^{m\tau}),
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\int_{\tau-r}^\tau e^{ms}\|\nabla^m u\|^2\,\mathrm{d}s\\
&\le\int_{\tau-r}^\tau e^{ms}\frac{H_1(t)}{\alpha}\,\mathrm{d}s+\int_{\tau-r}^\tau e^{ms}\frac{c_1}{\alpha}\,\mathrm{d}s\\
&\le\frac{1}{\alpha}\int_{\tau-r}^\tau e^{ms}H_1(t)\,\mathrm{d}s+\frac{c_1}{\alpha m}(e^{m\tau}-e^{m(\tau-r)}).
\end{aligned}\tag{3.17}$$

In the Bounded set $D \subset C_{V,H}$, for any $u \in D$, there exists a constant d such that

$$\|v\|^2 + \|\nabla^m u\|^2 \le d^2, \qquad (3.18)$$

$$H_1(t) = \|v\|^2 + \varepsilon^2 \|u\|^2 + \int_0^{\|\nabla^m u\|^2} (\phi(s) - \varepsilon\sigma(s))\,\mathrm{d}s \le d^2. \qquad (3.19)$$

We take a proper constant $\alpha > 0$, such that $-\gamma_1 + \frac{C_h^2 \lambda^{-m}}{\alpha\varepsilon} < 0$. And we can choose $m \in (0, m_0)$ small enough, such that $m - \gamma_1 + \frac{C_h^2 \lambda^{-m}}{\alpha\varepsilon} < 0$.

For this choice, by (3.16)-(3.19), (3.13) can be rewritten

$$\begin{aligned} e^{mt} H_1(t) &\le e^{m\tau} d^2 + \frac{1}{\varepsilon m}\|f\|^2 (e^{mt} - e^{m\tau}) + (m - \gamma_1 + \frac{C_h^2 \lambda^{-m}}{\alpha\varepsilon}) \int_\tau^t e^{ms} H_1(t)\,\mathrm{d}s \\ &\quad + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(e^{mt} - e^{m\tau}) + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(e^{m\tau} - e^{m(\tau - r)}) + \frac{C_h^2 \lambda^{-m} r}{\alpha\varepsilon}) e^{m\tau} d^2 \\ &\le e^{m\tau} d^2 (1 + \frac{C_h^2 \lambda^{-m} r}{\alpha\varepsilon}) + \frac{1}{\varepsilon m}\|f\|^2 (e^{mt} - e^{m\tau}) \\ &\quad + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(e^{mt} - e^{m\tau}) + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(e^{m\tau} - e^{m(\tau - r)}). \end{aligned} \qquad (3.20)$$

So, we can get by (3.20)

$$\begin{aligned} H_1(t) &\le e^{-mt} e^{m\tau} d^2 (1 + \frac{C_h^2 \lambda^{-m} r}{\alpha\varepsilon}) + \frac{1}{\varepsilon m}\|f\|^2 (1 - e^{-mt} e^{m\tau}) \\ &\quad + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(1 - e^{-mt} e^{m\tau}) + \frac{C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}(e^{m\tau} e^{-mt} - e^{m(\tau - r)} e^{-mt}) \\ &\le e^{-mt} e^{m\tau} d^2 (1 + \frac{C_h^2 \lambda^{-m} r}{\alpha\varepsilon}) + \frac{1}{\varepsilon m}\|f\|^2 + \frac{2 C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}. \end{aligned} \qquad (3.21)$$

If we denote

$$\rho_0^2 = \frac{1}{\varepsilon m}\|f\|^2 + \frac{2 C_h^2 c_1 \lambda^{-m}}{\alpha\varepsilon m}, \qquad \hat{\rho}_0^2 = 1 + \frac{C_h^2 \lambda^{-m} r}{\alpha\varepsilon},$$

then (3.21) yields that

$$\|v\|^2 + \varepsilon^2 \|u\|^2 + \int_0^{\|\nabla^m u\|^2} (\phi(s) - \varepsilon\sigma(s))\,\mathrm{d}s \le \rho_0^2 + \hat{\rho}_0^2 d^2 e^{m(\tau - t)}. \qquad (3.22)$$

By Assumption, we have

$$\|v\|^2 + \|\nabla^m u\|^2 \le \rho_0^2 + \hat{\rho}_0^2 d^2 e^{m(\tau - t)}, \qquad \forall t \ge \tau. \qquad (3.23)$$

What's more, before we prove uniqueness, we first prove a conclusion.

We denote

$$F(t) = f + h(t, u_t), \qquad t \geq t_0 - s. \qquad (3.24)$$

According to (G_3), we have

$$\|F(t)\| \leq \|f\| + L_h \|u_t\|_{C_H}. \qquad (3.25)$$

Then equation (2.1) become

$$\begin{aligned} &v' - \varepsilon v + \varepsilon^2 u + \sigma(\|\nabla^m u\|^2)(-\Delta)^m v - \varepsilon\sigma(\|\nabla^m u\|^2)(-\Delta)^m u \\ &+\phi(\|\nabla^m u\|^2)(-\Delta)^m u = F(t). \end{aligned} \qquad (3.26)$$

We use v multiply with both sides of equation (3.26) and obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon\|v\|^2 + \varepsilon^2(u,v) + \sigma(\|\nabla^m u\|^2)\|\nabla^m v\|^2 - \varepsilon\sigma(\|\nabla^m u\|^2)((-\Delta)^m u, v) \\ &+\phi(\|\nabla^m u\|^2)((-\Delta)^m u, v) = (F(t), v). \end{aligned} \qquad (3.27)$$

By using Hölder inequality, Young's inequality, we obtain

$$(F(t), v) \leq \|F(t)\|\|v\| \leq \frac{1}{4\varepsilon}\|F\|^2 + \varepsilon\|v\|^2. \qquad (3.28)$$

From (3.3), (3.5) and (3.28), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|v\|^2 + \varepsilon^2\|u\|^2 + \int_0^{\|\nabla^m u\|^2}(\phi(s) - \varepsilon\sigma(s))\,\mathrm{d}s] + \sigma(\|\nabla^m u\|^2)\|\nabla^m v\|^2 \\ &-2\varepsilon\|v\|^2 + \varepsilon^3\|u\|^2 + \varepsilon(\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^m u\|^2 \leq \frac{1}{4\varepsilon}\|F\|^2. \end{aligned} \qquad (3.29)$$

When assumption (A_1) hold, and $\varepsilon < \frac{\lambda^m}{2}$, we receive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|v\|^2 + \varepsilon^2\|u\|^2 + \int_0^{\|\nabla^m u\|^2}(\phi(s) - \varepsilon\sigma(s))\,\mathrm{d}s] + (\sigma(\|\nabla^m u\|^2) - 1)\|\nabla^m v\|^2 \\ &+(\lambda^m - 2\varepsilon)\|v\|^2 + \varepsilon^3\|u\|^2 + \varepsilon\int_0^{\|\nabla^m u\|^2}(\phi(s) - \varepsilon\sigma(s))\,\mathrm{d}s \leq \frac{1}{4\varepsilon}\|F\|^2. \end{aligned} \qquad (3.30)$$

Due to assumption, we get

$$\frac{d}{dt}H_2(t) + 2\gamma_2 H_2(t) + 2(\sigma(\|\nabla^m u\|^2) - 1)\|\nabla^m v\|^2 \leq \frac{1}{2\varepsilon}\|F\|^2, \qquad (3.31)$$

where $H_2(t)=\|v\|^2+\varepsilon^2\|u\|^2+\int_0^{\|\nabla^m u\|^2}(\phi(s)-\varepsilon\sigma(s))\,\mathrm{d}\,s,$

$\gamma_2=min\{2\lambda^m-4\varepsilon,2\varepsilon\}.$

Combining the Gronwall Lemma, we get

$$H_2(\mathrm{t}) < \mathrm{C}. \qquad (3.32)$$

At the same time, we also have

$$\int_0^t\|\nabla^m v\|^2\,\mathrm{d}\,s<C. \qquad (3.33)$$

Hence,

$$\begin{aligned}\int_s^t\|\nabla^m v\|^2\,\mathrm{d}\,s&\le(\int_s^t\|\nabla^m v\|^2\,\mathrm{d}\,s)^{\frac{1}{2}}(t-s)^{\frac{1}{2}}\\&\le\frac{k}{2}(t-s)+b,\qquad for\quad \forall t>s\ge 0,\quad \exists k,b>0.\end{aligned} \qquad (3.34)$$

Now, we prove the uniqueness of the solution. Assume that $u(\cdot)=u(\cdot;\tau,\psi)$ and $v(\cdot)=v(\cdot;\tau,\kappa)$ are the two solutions of the initial boundary value problem (1.1), ψ,κ are the corresponding initial value, we denote $w(\cdot)=u(\cdot)-v(\cdot)$. Therefore we have

$$\begin{aligned}&w''+\sigma(\|\nabla^m u\|^2)(-\Delta)^m u'-\sigma(\|\nabla^m v\|^2)(-\Delta)^m v'+\phi(\|\nabla^m u\|^2)(-\Delta)^m u\\&-\phi(\|\nabla^m v\|^2)(-\Delta)^m v=h(t,u_t)-h(t,v_t).\end{aligned} \qquad (3.35)$$

We take the inner product of the above equation with w' and we obtain

$$(w'',w')=\frac{1}{2}\frac{d}{dt}\|w'\|^2. \qquad (3.36)$$

ISSN 2059-9951

$$
\begin{aligned}
&(\sigma(\|\nabla^m u\|^2)(-\Delta)^m u' - \sigma(\|\nabla^m v\|^2)(-\Delta)^m v', w') \\
&= (\sigma(\|\nabla^m u\|^2)(-\Delta)^m u' - \sigma(\|\nabla^m u\|^2)(-\Delta)^m v' \\
&+\sigma(\|\nabla^m u\|^2)(-\Delta)^m v' - \sigma(\|\nabla^m v\|^2)(-\Delta)^m v', w') \\
&= \sigma(\|\nabla^m u\|^2)\|\nabla^m w'\|^2 + (\sigma(\|\nabla^m u\|^2 - \sigma(\|\nabla^m v\|^2)((-\Delta)^m v', w') \\
&= \sigma(\|\nabla^m u\|^2)\|\nabla^m w'\|^2 + \sigma'(\xi)(\|\nabla^m u\| + \|\nabla^m v\|)\|\nabla^m w\|((-\Delta)^m v', w') \qquad (3.37) \\
&\ge \sigma(\|\nabla^m u\|^2)\|\nabla^m w'\|^2 - \|\sigma'(\xi)\|_\infty (\|\nabla^m u\| + \|\nabla^m v\|)\|\nabla^m v'\|\|\nabla^m w\|\|\nabla^m w'\| \\
&= \sigma(\|\nabla^m u\|^2)\|\nabla^m w'\|^2 - c_2\|\nabla^m w\|\|\nabla^m w'\| \\
&\ge \sigma(\|\nabla^m u\|^2)\|\nabla^m w'\|^2 - \frac{\varepsilon}{2}\|\nabla^m w\|^2 - \frac{c_2^2}{2\varepsilon}\|\nabla^m w'\|^2 \\
&= (\sigma(\|\nabla^m u\|^2) - \frac{c_2^2}{2\varepsilon})\|\nabla^m w'\|^2 - \frac{\varepsilon}{2}\|\nabla^m w\|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(\phi(\|\nabla^m u\|^2)(-\Delta)^m u - \phi(\|\nabla^m v\|^2)(-\Delta)^m v, w') \\
&\ge \frac{1}{2}\phi(\|\nabla^m u\|^2)\frac{d}{dt}\|\nabla^m w\|^2 - \frac{\varepsilon}{2}\|\nabla^m w\|^2 - \frac{c_3^2}{2\varepsilon}\|\nabla^m w'\|^2.
\end{aligned} \qquad (3.38)
$$

Therefore, by the above inequality

$$
\begin{aligned}
&\frac{d}{dt}\|w'\|^2 + \phi(\|\nabla^m u\|^2)\frac{d}{dt}\|\nabla^m w\|^2 + (2\sigma(\|\nabla^m u\|^2) - \frac{c_2^2 + c_3^2}{\varepsilon})\|\nabla^m w'\|^2 \\
&-\varepsilon\|\nabla^m w\|^2 \le 2(h(t,u_t) - h(t,v_t), w').
\end{aligned} \qquad (3.39)
$$

Since

$$
2(h(t,u_t) - h(t,v_t), w') \le \|h(t,u_t) - h(t,v_t)\|^2 + \|w'\|^2. \qquad (3.40)
$$

According to (A_2), and $\sigma > \frac{c_2^2 + c_3^2}{2\varepsilon}$, we have

$$
\frac{d}{dt}(\|w'\|^2 + \mu\|\nabla^m w\|^2) \le \|h(t,u_t) - h(t,v_t)\|^2 + \|w'\|^2 + \varepsilon\|\nabla^m w\|^2. \qquad (3.41)
$$

So,

$$\begin{aligned}&\frac{d}{dt}(\|w'\|^2+\|\nabla^m w\|^2)\\&\le\frac{1}{\beta}\|h(t,u_t)-h(t,v_t)\|^2+\frac{1}{\beta}\|w'\|^2+\frac{\varepsilon}{\beta}\varepsilon\|\nabla^m w\|^2\\&\le\frac{1}{\beta}\|h(t,u_t)-h(t,v_t)\|^2+c_4(\|w'\|^2+\|\nabla^m w\|^2),\end{aligned}\tag{3.42}$$

where $\beta=min\{1,\mu\}$, $c_4=max\{\frac{1}{\beta},\frac{\varepsilon}{\beta}\}$.

Due to

$$\begin{aligned}&\int_\tau^t\|h(t,u_t)-h(t,v_t)\|^2\,\mathrm{d}s\\&\le C_h^2\int_{\tau-r}^{\tau}\|u-v\|^2\,\mathrm{d}s\\&\le\lambda^{-m}C_h^2r\|\psi-\kappa\|^2_{C_{V,H}}+\lambda^{-m}C_h^2\int_\tau^t\|w\|^2\,\mathrm{d}s\end{aligned}\tag{3.43}$$

Integrating (3.42)over the interval $[\tau,t]$, we can get

$$\begin{aligned}&\|w'(t)\|^2+\|\nabla^m w(t)\|^2\\&\le\|w'(\tau)\|^2+\|\nabla^m w(\tau)\|^2+c_4\int_\tau^t(\|w'\|^2+\|\nabla^m w(\tau)\|^2)\,\mathrm{d}s\\&+\beta^{-1}\lambda^{-m}C_h^2r\|\psi-\kappa\|^2_{C_{V,H}}+\beta^{-1}\lambda^{-m}C_h^2\int_\tau^t\|w\|^2\,\mathrm{d}s\\&=(1+\beta^{-1}\lambda^{-m}C_h^2r)\|\psi-\kappa\|^2_{C_{V,H}}+\int_\tau^t((c_4+\beta^{-1}\lambda^{-m}C_h^2)\|\nabla^m w\|^2+c_4\|w'\|^2)\,\mathrm{d}s.\end{aligned}\tag{3.44}$$

Set $\gamma_3=max\{c_4+\beta^{-1}\lambda^{-m}C_h^2,c_4\}$, then we have

$$\begin{aligned}&\|w'(t)\|^2+\|\nabla^m w(t)\|^2\\&\le(1+\beta^{-1}\lambda^{-m}C_h^2r)\|\psi-\kappa\|^2_{C_{V,H}}+\gamma_3\int_\tau^t(\|\nabla^m w\|^2+\|w'\|^2)\,\mathrm{d}s.\end{aligned}\tag{3.45}$$

Combining the Gronwall Lemma, we get

$$\begin{aligned}&\|w'(t)\|^2+\|\nabla^m w(t)\|^2\\&\le(1+\beta^{-1}\lambda^{-m}C_h^2r)\|\psi-\kappa\|^2_{C_{V,H}}\,e^{\gamma_3(t-\tau)},\quad for\,all\quad t\ge\tau.\end{aligned}\tag{3.46}$$

If ψ and κ stand for the same initial value, there has

$$\|w'(t)\|^2+\|\nabla^m w(t)\|^2\le 0.\tag{3.47}$$

That shows that

$$\|w'(t)\|^2 = 0, \qquad \|\nabla^m w(t)\|^2 = 0. \qquad (3.48)$$

That is

$$\mathrm{w}(\mathrm{t}) = 0. \qquad (3.49)$$

Therefore

$$\mathrm{u} = \mathrm{v}. \qquad (3.50)$$

We get the uniqueness of the solution. So the proof of the theorem 3.1 has been completed.

4 EXISTENCE OF THE PULLBACK ATTRACTOR

In this subsection, we assume that $f \in H$, we aim to study the pullback attractor for the initial value problem (1.1).

From Theorem 3.1, the initial value problem (1.1) generates a family

two-parameter semigroup $U(\cdot,\cdot)$ in $C_{V,H}$, which can be defined by

$$U(t,\tau)(\psi) = u_t(\cdot,;\tau,\psi), \quad t \ge \tau, \quad \psi \in C_{V,H}. \qquad (4.1)$$

Lemma 4.1 Let $\psi, \kappa \in C_{V,H}$ be the two initial values for the problem (1.1), and $\tau \in R$ is the initial time. Denote by $u(\cdot) = u(\cdot;\tau,\psi)$ and $v(\cdot) = v(\cdot;\tau,\kappa)$ the corresponding solution to (1.1). Then, there exists a constant γ_3 which does not depend on the initial data and time, such that

$$\begin{aligned} &\|u'(t) - v'(t)\|^2 + \|\nabla^m u(t) - \nabla^m v(t)\|^2 \\ &\le (1 + \beta^{-1}\lambda^{-m} C_h^2 r)\|\psi - \kappa\|_{C_{V,H}}^2 e^{\gamma_3 (t-\tau)}, \quad \forall t \ge \tau. \end{aligned} \qquad (4.2)$$

and

$$\|u_t - v_t\|_{C_{V,H}}^2 \le (1 + \beta^{-1}\lambda^{-m} C_h^2 r)\|\psi - \kappa\|_{C_{V,H}}^2 e^{\gamma_3 (t-\tau)}, \quad \forall t \ge \tau + r. \qquad (4.3)$$

Proof. We denote $w = u - v$, by (3.35), we can get (4.2) easily.

If we consider $t \ge \tau + r$, then $t + \theta \ge \tau$ for any $\theta \in [-r, 0]$, and

$$\begin{aligned}
&\|w'(t+\theta)\|^2+\|\nabla^m w(t+\theta)\|^2 \\
&\le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau+\theta)} \qquad (4.4)\\
&\le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau)}, \quad \forall t\ge\tau+r.
\end{aligned}$$

Thus,

$$\|w_t\|^2 \le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau)}, \quad \forall t\ge\tau+r. \qquad (4.5)$$

Theorem 4.1 The mapping $U(t,\tau):C_{V,H}\to C_{V,H}$ is continuous for any $t\ge\tau$.

Proof. Let $\psi,\kappa\in C_{V,H}$ be the initial value for the problem (1.1) and $t\ge\tau$. Denote by $u(\cdot)=u(\cdot;\tau,\psi)$ and $v(\cdot)=v(\cdot;\tau,\kappa)$ the corresponding solution to (1.1). Then, writing again $w=u-v$ we obtain the following. If $t\in[\tau-r,\tau]$, then $w(t)=\psi(t-\tau)-\kappa(t-\tau)$ and

$$\begin{aligned}
&\|w'(t)\|^2+\|\nabla^m w(t)\|^2 \\
&\le \|\psi-\kappa\|_{C_V}^2+\|\psi'-\kappa'\|_{C_H}^2 \qquad (4.6)\\
&\le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau+r)}.
\end{aligned}$$

Thus, we have

$$\|w'(t)\|^2+\|\nabla^m w(t)\|^2 \le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau+r)}, \quad \forall t\ge\tau-r, \qquad (4.7)$$

Whence

$$\|w_t\|^2 \le (1+\beta^{-1}\lambda^{-m}C_h^2 r)\|\psi-\kappa\|_{C_{V,H}}^2 e^{\gamma_3(t-\tau+r)}, \quad \forall t\ge\tau, \qquad (4.8)$$

which implies the continuity of $U(t,\tau)$.

Theorem 4.2 Assume that condition (A_1)-(A_2) and (G_1)-(G_4) hold with $m_0>0$, and that $f\in H$. Suppose in addition that $C_h^2\lambda^{-m}<\gamma_1\alpha\varepsilon$. Then, there exists a family $\{B(t)\}_{t\in R}$ of bounded sets in $C_{V,H}$ which is uniformly pullback absorbing for the process $U(\cdot,\cdot)$. Moreover, $B(t)=B^0$ for all $t\in R$, where B^0 is a bounded set in $C_{V,H}$.

Proof. By (3.23), we can have

$$\|u'(t;\tau,\psi)\|^2+\|\nabla^m u(t;\tau,\psi)\|^2 \le \rho_0^2+\hat\rho_0^2 d^2 e^{m(\tau-t)}, \qquad \forall t\ge\tau. \qquad (4.9)$$

and

$$\|u'(t;\tau,\psi)\|^2 + \|\nabla^m u(t;\tau,\psi)\|^2 \le \rho_0^2 + \hat{\rho}_0^2 d^2, \qquad \forall t \ge \tau. \qquad (4.10)$$

Moreover, as $u(t;\tau,\psi) = \psi(t-\tau)$ and $u'(t;\tau,\psi) = \psi'(t-\tau)$ for $t \in [\tau - r, \tau]$, then inequality (4.10) holds true for $t \ge \tau - r$.

If we take now $t \ge \tau + r$, then for all $\theta \in [-r, 0]$ we have $t + \theta \ge \tau$ and so

$$\|u'(t+\theta;\tau,\psi)\|^2 + \|\nabla^m u(t+\theta;\tau,\psi)\|^2 \le \rho_0^2 + \hat{\rho}_0^2 d^2 e^{m(\tau-t)} e^{mr}, \qquad \forall t \ge \tau. \quad (4.11)$$

or, in other words,

$$\|U(t,\tau)\psi\|_{C_{V,H}}^2 \le \rho_0^2 + \hat{\rho}_0^2 d^2 e^{m(\tau-t)} e^{mr}, \qquad \forall t \ge \tau + r, \quad \psi \in D. \qquad (4.12)$$

Therefore, there exists $T_D \ge r$ such that

$$\|U(t,t-s)\psi\|_{C_{V,H}}^2 \le \rho_0^2, \qquad \forall t \in R, \quad s \in T_D, \quad \psi \in D. \qquad (4.13)$$

which means that the ball $B_{C_{V,H}}(0,\rho_0) = B^0 \subset C_{V,H}$ is uniformly pullback absorbing for the process $U(\cdot,\cdot)$.

Remark On the one hand, observe that if $t_0 \in R$ and $t \ge t_0$, then

$$u(t+\theta;t_0-s,\psi) = u(t+\theta;t-(s+t-t_0),\psi)$$

and

$$u'(t+\theta;t_0-s,\psi) = u'(t+\theta;t-(s+t-t_0),\psi)$$

with $s+t-t_0 \ge s$. As a consequence of (4.13) we have

$$\|U(t,t_0-s)\psi\|_{C_{V,H}}^2 \le \rho_0^2, \quad \forall t_0 \in R, \quad t \ge t_0, \quad s \in T_D, \quad \psi \in D. \quad (4.14)$$

or equivalently, we have $\forall t_0 \in R, t \ge t_0, \theta \in [-r,0], s \in T_D, \psi \in D,$

$$\|u'(t+\theta;t_0-s,\psi)\|^2 + \|\nabla^m u(t+\theta;t_0-s,\psi)\|^2 \le \rho_0^2. \qquad (4.15)$$

On the other hand, (4.10) implies, $\forall t_0 \in R, t \ge t_0, s \in R, t \in t_0 - s - r, \psi \in D,$

$$\|u'(t+\theta;t_0-s,\psi)\|^2 + \|\nabla^m u(t+\theta;t_0-s,\psi)\|^2 \le \rho_0^2 + \hat{\rho}_0^2 d^2. \qquad (4.16)$$

Lemma 4.2 [15] Let $y: R^+ \to R^+$ be an absolutely continuous function satisfying:

$$\frac{d}{dt}y(t)+2\varepsilon y(t)\le h(t)y(t)+z(t), \quad t>0, \qquad (4.17)$$

where $\varepsilon>0, z\in L^1_{loc}(R^+), \int_s^t h(s)\,\mathrm{d}s\le \varepsilon(t-s)+m$ for $t\ge s\ge 0$, and $\exists m>0$, then

$$y(t)\le e^m(y(0)e^{-\varepsilon t}+\int_0^t |z(t)|\ e^{-\varepsilon(t-\tau)}\,\mathrm{d}\tau), \quad t>0, \qquad (4.18)$$

Lemma 4.3 In addition to the assumptions in Theorem 4.1, suppose that condition (G_5) holds. Then, there exists a compact set $B_2\subset C_{V,H}$ which is uniformly pullback attracting for the process $U(\cdot,\cdot)$, and consequently, there exists the pullback attractor $A(t)_{t\in R}$. Moreover, $A(t)_{t\in R}\subset C_{D(A),V}$ for all $t\in R$.

Proof. For each $\varepsilon\in R$, the norm $\|\psi\|_\varepsilon^2=\|\psi\|_{C_V}^2+\|\psi'+\varepsilon\psi\|_{C_H}^2$, $\psi\in C_{V,H}$ is equivalent to $\|\cdot\|_0 :=\|\cdot\|_{C_{V,H}}$. This allows us to obtain absorbing ball for the original norm by proving the existence of absorbing balls for this new norm for some suitable value of ε. Indeed,let us denote

$B_\varepsilon(0,\rho)=\{\psi\in C_{V,H}:\|\psi\|_\varepsilon<\rho\}$.

Noticing that for $c_5=max\{2,1+2\varepsilon^2\lambda^{-m}\}$ it follows that

$$\begin{aligned}\|\psi\|_{C_{V,H}}^2&=\|\psi\|_{C_V}^2+\|\psi'+\varepsilon\psi-\varepsilon\psi\|_{C_H}^2\\&\le\|\psi\|_{C_V}^2+2\|\psi'+\varepsilon\psi\|_{C_H}^2+2\varepsilon^2\|\psi\|_{C_H}^2\\&\le(1+2\varepsilon^2\lambda^{-m})\|\psi\|_{C_V}^2+2\|\psi'+\varepsilon\psi\|_{C_H}^2\\&\le c_5\|\psi\|_\varepsilon^2.\end{aligned} \qquad (4.19)$$

then we have $B_\varepsilon(0,\rho)\subset B_0(0,c_5^{\frac{1}{2}}\rho)$.

Let $D\subset C_{V,H}$ be a bounded set, i.e. there exists $d>0$ such that for any $\psi\in D$ it holds $\|\psi\|_\varepsilon^2=\|\psi\|_{C_V}^2+\|\psi'+\varepsilon\psi\|_{C_H}^2\le d^2$, and so, $\|\psi\|_{C_{V,H}}^2\le c_5 d^2$.

Denote, as usual, by $u(\cdot)=u(\cdot;\tau,\psi)$ the solution of problem (2.1), and consider the following problems.

$$\begin{cases} v'' + \sigma(\| \nabla^m u\|^2)(-\Delta)^m v' + \phi(\| \nabla^m u\|^2)(-\Delta)^m v = f(t) + h(t,u_t),\ t \geq \tau, \\ v(t) = 0,\ t \in [\tau - r, \tau], \\ v'(t) = 0,\ t \in [\tau - r, \tau], \end{cases} \tag{4.20}$$

$$\begin{cases} w'' + \sigma(\| \nabla^m u\|^2)(-\Delta)^m w' + \phi(\| \nabla^m u\|^2)(-\Delta)^m w = 0,\ t \geq \tau, \\ w(t) = \psi(t-\tau),\ t \in [\tau - r, \tau], \\ w'(t) = \psi'(t-\tau),\ t \in [\tau - r, \tau], \end{cases} \tag{4.21}$$

From the uniqueness of the solution of problem (2.1), (4.20) and (4.21) it follows that

$$u(\cdot) = v(\cdot) + w(\cdot), \quad \forall t \in R, and\ \forall t \geq \tau - r. \tag{4.22}$$

Consequently, $U(t,\tau)$ can be written as

$$U(t,\tau)(\psi) = U_1(t,\tau)(\psi) + U_2(t,\tau)(\psi), \quad \psi \in C_{V,H}, \quad t \geq \tau - r. \tag{4.23}$$

where $U_1(t,\tau)(\psi) = v_t(\cdot) = v_t(\cdot;\tau,\psi)$ and $U_2(t,\tau)(\psi) = w_t(\cdot) = w_t(\cdot;\tau,\psi)$ are the solution of (4.20) and (4.21) respectively.

First,thanks to (4.10), but with $f = h = 0$, it follows

$$\|w'(t;\tau,\psi)\|^2 + \|\nabla^m w(t;\tau,\psi)\|^2 \leq c_5 d^2, \quad \forall t \geq \tau, \quad \psi \in D, \tag{4.24}$$

and by means of (4.11),

$$\|w_t(\cdot;\tau,\psi)\|_{C_V}^2 + \|w_t'(\cdot;\tau,\psi)\|^2 \leq c_5 d^2 e^{mr} e^{m(\tau-t)}, \quad \forall t \geq \tau + r, \quad \psi \in D, \tag{4.25}$$

Furthermore, for $t_0 \in R$, $t \geq t_0$ and $s > T_D \geq r$,

$$w(t;t_0 - s,\psi) = w(t;t-(s+t-t_0),\psi), \tag{4.26}$$

with $s + t - t_0 > s \geq T_D \geq r$.

Thus, (4.25) implies in particular

$$\begin{aligned} \|w(t;t_0 - s,\psi)\|^2 &\leq c_5 d^2 e^{mr} e^{m(t_0 - s - t)} \\ &\leq c_5 d^2 e^{mr} e^{-ms}, \quad \forall t_0 \in R, \quad t \geq t_0, \quad s \geq T_D, \quad \psi \in D. \end{aligned} \tag{4.27}$$

Then, (4.25) yields that

$$\|U_2(t,t-s)\psi\|_{C_{V,H}}^2 \leq c_5 d^2 e^{mr} e^{-ms}, \quad \forall t \in R, \quad s \geq r, \quad \psi \in D. \tag{4.28}$$

Whence

$$\lim_{s\to+\infty}\sup_{t\in R}\sup_{\psi\in D}\|U_2(t,t-s)\psi\|^2_{C_{V,H}}=0. \qquad (4.29)$$

Let us now proceed with the other term. Let us fix $t_0\in R$, $s\geq T_D$, $\psi\in D$ and denote

$$u(t)=u(t;t_0-s,\psi),\quad v(t)=v(t;t_0-s,\psi),\quad t\geq t_0-s-r, \qquad (4.30)$$

and

$$F(t)=f+h(t,u_t),\quad t\geq t_0-s. \qquad (4.31)$$

Then

$$\|F(t)\|\leq\|f\|+L_h\|u_t\|_{C_H}. \qquad (4.32)$$

From (4.15) we obtain

$$\|F(t)\|\leq\|f\|+L_h\lambda^{\frac{-m}{2}}\rho_0=L_1,\quad \forall t\geq t_0. \qquad (4.33)$$

and from (4.16)

$$\begin{aligned}\|F(t)\|&\leq\|f\|+L_h\lambda^{\frac{-m}{2}}(\rho_0^2+\hat{\rho}_0^2d^2)^{\frac{1}{2}}\\&\leq L_1+L_h\lambda^{\frac{-m}{2}}\rho_0 d,\quad \forall t\geq t_0-S.\end{aligned} \qquad (4.34)$$

Let $q=v'+\varepsilon v$, then equation (4.20) become

$$\begin{aligned}&q'-\varepsilon q+\varepsilon^2 v+\sigma(\|\nabla^m u\|^2)(-\Delta)^m q-\varepsilon\sigma(\|\nabla^m u\|^2)(-\Delta)^m v\\&+\phi(\|\nabla^m u\|^2)(-\Delta)^m v=F(t).\end{aligned} \qquad (4.35)$$

We use $(-\Delta)q$ multiply with both sides of equation (4.35) and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla q\|^2+\frac{\varepsilon^2}{2}\frac{d}{dt}\|\nabla v\|^2+\frac{1}{2}(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\frac{d}{dt}\|\nabla^{m+1}v\|^2\\&+\varepsilon^3\|\nabla v\|^2+\varepsilon(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2\\&+\sigma(\|\nabla^m u\|^2)\|\nabla^{m+1}q\|^2\leq\frac{1}{2}\|F\|^2+\frac{1}{2}\|\Delta q\|^2.\end{aligned} \qquad (4.36)$$

Among (4.36), we have

$$
\begin{aligned}
&(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\frac{d}{dt}\|\nabla^{m+1}v\|^2\\
&=\frac{d}{dt}[(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2]\\
&-2\|\nabla^{m+1}v\|^2(\phi'(\|\nabla^m u\|^2)-\varepsilon\sigma'(\|\nabla^m u\|^2))(\nabla^m u,\nabla^m u') \qquad (4.37)\\
&\geq\frac{d}{dt}[(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2]\\
&-2\|\nabla^{m+1}v\|^2(\phi'(\|\nabla^m u\|^2)-\varepsilon\sigma'(\|\nabla^m u\|^2))\|\nabla^m u\|\|\nabla^m u'\|
\end{aligned}
$$

$$
\begin{aligned}
&\sigma(\|\nabla^m u\|^2)\|\nabla^{m+1}q\|^2-\frac{1}{2}\|\Delta q\|^2\\
&\geq(\sigma(\|\nabla^m u\|^2)-1)\|\nabla^{m+1}q\|^2 \qquad (4.38)\\
&\geq\lambda^m(\sigma(\|\nabla^m u\|^2)-1)\|\nabla q\|^2.
\end{aligned}
$$

Substitution (4.37), (4.38) into (4.36)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\nabla q\|^2+\varepsilon^2\|\nabla v\|^2+(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2]+\varepsilon^3\|\nabla v\|^2\\
&+\lambda^m(\sigma(\|\nabla^m u\|^2)-1)\|\nabla q\|^2+\varepsilon(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2 \qquad (4.39)\\
&\leq\frac{1}{2}\|F\|^2+\|\nabla^{m+1}v\|^2(\phi'(\|\nabla^m u\|^2)-\varepsilon\sigma'(\|\nabla^m u\|^2))\|\nabla^m u\|\|\nabla^m u'\|
\end{aligned}
$$

That is,

$$
\begin{aligned}
\frac{d}{dt}y(t)+\gamma_4 y(t)&\leq\|F\|^2+C\|\nabla^m u'\|\|\nabla^{m+1}v\|^2 \qquad (4.40)\\
&\leq\|F\|^2+C\|\nabla^m u'\|y(t).
\end{aligned}
$$

where

$$y(t)=\|\nabla q\|^2+\varepsilon^2\|\nabla v\|^2+(\phi(\|\nabla^m u\|^2)-\varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1}v\|^2,$$

$$\gamma_4=min\{2(\sigma(\|\nabla^m u\|^2)-1),2\varepsilon\}.$$

On the one hand, for all $t\geq t_0-s$

$$\frac{d}{dt}y(t)+\gamma_4 y(t)\leq(L_1+L_h\lambda^{\frac{-m}{2}}\rho_0 d)^2+C\|\nabla^m u'\|y(t). \qquad (4.41)$$

Noticing that $y(t_0-s)=0$,

According to Lemma 4.2 and (3.34), we obtain

$$y(t_0) \le Cy(t_0 - s)e^{-kt_0} + \int_{t_0-s}^{t_0} e^{-ks} \|F\|^2 \, \mathrm{d}t$$
$$\le \|F\|^2 = (L_1 + L_h \lambda^{\frac{-m}{2}} \rho_0 d)^2 = L_2^2 \qquad (4.42)$$

On the other hand, if $t \ge t_0$, once again,

$$y(t) \le Cy(t_0)e^{-kt} + \int_{t_0}^{t} e^{k(t-t_0)} \|F\|^2 \, \mathrm{d}t$$
$$\le \|F\|^2 + Cy(t_0)e^{-kt} \qquad (4.43)$$
$$\le L_1^2 + CL_2^2 e^{-kt}$$

Then, there exists $T_D' \ge T_D$ such that, if $s \ge T_D'$,

$$y(t) \le L_1^2 + CL_2^2 e^{-kt}, \quad t_0 \in R, \quad t \ge t_0. \qquad (4.44)$$

Recall that $y(t) = y(t; t_0 - s, \psi)$, if we fix $t \ge t_0$, take $s = T_D'$ and denote $\tilde{s} = t - t_0 + T_D'$ we have, provided t is large enough, that

$$y(t; t_0 - T_D', \psi) = y(t; t - (t - t_0 + T_D'), \psi) = y(t; t - \tilde{s}, \psi) \le 2L_1^2. \qquad (4.45)$$

In conclusion, there exists $T_D'' > 0$ such that for all $t \in R$, and all $s \ge T_D' + T_D''$,

$$y(t; t - s, \psi) \le 2L_1^2, \quad \forall \psi \in D. \qquad (4.46)$$

Denoting $\hat{T}_D = T_D' + T_D'' + R$, we have for all $\psi \in D$, $t \in R$, $s \ge \hat{T}_D$

$$\|\nabla q\|^2 + \varepsilon^2 \|\nabla v\|^2 + (\phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2))\|\nabla^{m+1} v\|^2 \le 2L_1^2, \qquad (4.47)$$

where $F(t; t - s, \psi) = f + h(t, u_t(\cdot; t - s, \psi))$.

Consequently, for all $\psi \in D$, $t \in R$, $s \ge \hat{T}_D$,

$$\|\nabla v'\|^2 + \|\nabla^{m+1} v\|^2 \le \frac{2}{\gamma_5} L_2^2 \le \frac{4}{\gamma_5} \|f\|^2 + \frac{4}{\gamma_5} L_h^2 \lambda^{-m} \rho_0^2, \qquad (4.48)$$

where $\gamma_5 = min\{1, \varepsilon^2, \phi(\|\nabla^m u\|^2) - \varepsilon\sigma(\|\nabla^m u\|^2)\}$, and by repeating once more the same argument previously used,

$$\|\nabla v_t(\cdot; t - s, \psi)\|^2_{C_{D(A), H_0^1}} \le \rho_1^2 = \frac{4}{\gamma_5} \|f\|^2 + \frac{4}{\gamma_5} L_h^2 \lambda^{-m} \rho_0^2,, \qquad (4.49)$$

for all $\psi \in D$, $t \in R$, $s \ge \hat{T}_D$.

This means that the ball $B^1 = B_{C_{D(A),H_0^1}}(0,\rho_1)$ is bounded set in $C_{D(A),H_0^1}$ which, in addition, is uniformly pullback absorbing for the family of operators $U_1(\cdot,\cdot)$. As B^1 is a bounded set in $C_{V,H}$, then there exists $T_{B^1} \geq r$ such that

$$U_1(t,t-s)B^1 \subset B^1, \quad \forall t \in R, \quad s \geq T_{B^1}, \qquad (4.50)$$

and, therefore, the bounded set $B^2 \subset C_{D(A),H_0^1}$ given by

$$B^2 = \bigcup_{t\in R} \bigcup_{s\geq T_{B^1}} U_1(t,t-s)B^1 \subset B^1 \subset B^1, \qquad (4.51)$$

is uniformly pullback absorbing for $U_1(\cdot,\cdot)$ in $C_{V,H}$.

By Ascoli-Arzelà theorem, we can prove that $\overline{B^2}$ is compact, so

$\{B(t) \equiv \overline{B^2}\}_{t\in R}$ is a family of compact subsets in $C_{V,H}$, which is also uniformly pullback attracting for $U(\cdot,\cdot)$, and the proof has been completed.

5 CONCLUSIONS

The paper's main results deal with the pullback attractors of the Higher-order Kirchhoff-type equation (1.1) with strong nonlinear damping and delays. In section 2, we introduce some basic concepts. Under of the proper assume, in section 3, we prove the existence and uniqueness of the solution by Galerkin method. At last, in section 4, we obtain the existence of the pullback attractor. The technology we use is introduced in [6], that is, we divide the semigroup into two: the one is a asymptotically close to 0,while the other is uniformly compact, so we can get the pullback attractor.

6 ACKNOWLEDFEMENTS

We express our sincere thanks to the anonymous reviewer for his/her careful reading of the paper, we hope that we can get valuable comments and suggestions. These contributions greatly improved the paper, and making the paper better.

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