THE FINITE HAUSDORFF AND FRACTAL DIMENSIONS OF THE GLOBAL ATTRACTOR FOR A CLASS KIRCHHOFF-TYPE EQUATIONS

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ABSTRACT

In this paper, we study the Hausdorff and Fractal dimensions of the global attractor for a class Kirchhoff-type equations with strongly damped terms and source terms. We testify the dynamical system associated with above-mentioned models is Frechet differential under suitable conditions. Furthermore, we obtain a precise estimate of Hausdorff and Fractal dimensions of the global attractor.

Keywords: Kirchhoff-type equations; The global attractor; Frechet differential; Hausdorff and Fractal dimensions.
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1. INTRODUCTION

Guoguang Lin and Xiangshuang Xia [1] studied the existence of a global attractor for a Kirchhoff type equations with strongly damped terms and source terms. Furthermore, In this paper, we consider the finite Hausdorff and Fractal dimensions of the global attractor for the above mentioned models as following:

$$u_{tt} - M(\|\nabla u\|^2 + \|\nabla^m v\|^2)\Delta u - \beta\Delta u_t + g_1(u,v) = f_1(x), \quad (1.1)$$

$$v_{tt} + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(-\Delta)^m v + \beta(-\Delta)^m v_t + g_2(u,v) = f_2(x), \quad (1.2)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (1.3)$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x), \quad x \in \Omega, \quad (1.4)$$

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \quad (1.5)$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\beta > 0$ is a constant and $f_i(x)(i=1,2)$ is a given source term. Moreover, $M(\|\nabla u\|^2 + \|\nabla^m v\|^2)$ is a scalar function. Then the assumptions on M and $g_i(u,v)$ will be specified later.

Generally speaking, the Hausdorff dimensions has been studied by some authors .In 1990s, S Zhou[2]
obtained a more precise estimate of the upper bound of the Hausdorff dimension of attractor for strongly damped nonlinear wave equations,

$$\begin{aligned} &u_{tt} - \alpha\Delta u_t - \Delta u + h(u_t) + f(u) = g, \quad x \in \Omega, t > 0, \\ &u(x,t)\big|_{x\in\partial\Omega} = 0, \quad t > 0, \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega. \end{aligned} \quad (1.6)$$

Meanwhile, He also obtained Hausdorff dimension decreases as the strong damping grew for large damping. Later, Xiaoming Fan and S Zhou [3] got the Hausdorff dimension of Kernel sections, which

decreases as the strong damping grew for large strong damping under some conditions, particularly in the autonomous case.
In recently years, the studies about the estimate of dimensions is developing. In 2014, Meixia Wang, Cuicui Tian and Guoguang Lin [4] got the upper bound estimation of the Hausdorff and Fractal dimensions of attractor.

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u_x^2 = 0 . \qquad (1.7)$$

In 2017, Yunlong Gao[5] considered the Hausdorff dimension of a global attractor for a class of strongly damped Higher-order Kirchhoff type equations:

$$u_{tt} + (-\Delta)^m u_t + (\alpha + \beta\|\nabla^m u\|^2)^q(-\Delta)^m u + g(u) = f(x), \quad (x,t) \in \Omega\times[0,+\infty), \qquad (1.8)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \qquad (1.9)$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, \quad i = 1,2,\cdots,m-1, x \in \partial\Omega, t \in (0,+\infty). \qquad (1.10)$$

Thanks to the previous results of the Hausdorff, we investigate the Hausdorff dimension of the high and low order coupled Kirchhoff-type equations. Firstly, we obtain the solution's semigroup is Frechet differential. Then, we establish a more precise estimate of the Hausdorff and Fractal dimensions.

2. The Estimates of the Hausdorff Dimensions for the Global Attractor.

In this paper, some inner product, norms, abbreviations and some assumptions $(H_1)-(H_6)$ and notations needed in the proof of our results is refer to[1].

2.1. Differentiability of the semigroup.

In order to estimate dimensions, we assume: (H_7): there exists μ_0, μ_1, such that

$$0 < \mu_0 \le M(\|\nabla u_2\|^2 + \|\nabla^m v_2\|^2) \le \mu_1, \qquad \mu = \begin{cases} \mu_0 & \frac{d}{dt}(\|\nabla\theta_1\|^2 + \|\nabla^m\theta_2\|^2) > 0 \\ \mu_1 & \frac{d}{dt}(\|\nabla\theta_1\|^2 + \|\nabla^m\theta_2\|^2) < 0 \end{cases}. \qquad (2.1)$$

and $M(\|\nabla u\|^2 + \|\nabla^m v\|^2) \in C^2(\Omega)$.

(H_8): For every $M > 0$, there exist $k = k(M)$, such that :

$$\|g_{iu}(\alpha u + (1-\alpha)u, v) - g_{iu}(u,v)\| \le k(\|\nabla u - \nabla u\|^{\delta_1} + \|\nabla^m v - \nabla^m \tilde{v}\|^{\delta_1}), \qquad (2.2)$$

$$\|g_{iv}(u, \alpha\tilde{v} + (1-\alpha)v) - g_{iv}(u,v)\| \le k(\|\nabla u - \nabla u\|^{\delta_1} + \|\nabla^m v - \nabla^m \tilde{v}\|^{\delta_1}), \qquad (2.3)$$

for any $u, u, \in H_0^1(\Omega); v, \tilde{v} \in H_0^m(\Omega); \|\nabla u\|, \|\nabla u\| \le M, \|\nabla v\|, \|\nabla\tilde{v}\| \le M$, and $\alpha \in (0,1), \delta_1 > 1$

The inner product and the norm in E_0 space are defined as follows:
$\forall \varphi_i = (u_i, v_i, p_i, q_i) \in E_0, (i = 1,2)$, we have

$$(\varphi_1, \varphi_2)_{E_0} = (\nabla u_1, \nabla u_2) + (\nabla^m v_1, \nabla^m v_2) + (p_1, p_2) + (q_1, q_2), \qquad (2.4)$$

$$\|\varphi_1\|_{E_0}^2 = \|\nabla u_1\|^2 + \|\nabla^m v_1\|^2 + \|p_1\|^2 + \|q_1\|^2 . \qquad (2.5)$$

Setting $\forall\varphi=(u,v,p,q)^T\in E_0,\ p=u_t+\varepsilon u, q=v_t+\varepsilon v,\ 0<\varepsilon<\min\{1,\frac{\lambda_1(3-2\beta)}{2},\frac{\lambda_1{}^m(3-2\beta)}{2},$ $\frac{-5-2\lambda_1\beta+\sqrt{(5+2\lambda_1\beta)^2+16\lambda_1\beta}}{4},\frac{-5-2\lambda_1{}^m\beta+\sqrt{(5+2\lambda_1{}^m\beta)^2+16\lambda_1{}^m\beta}}{4}\}$, the Eq. (1.1)-(1.5) is
equivalent to

$$\varphi_t+H(\varphi)=F(\varphi), \tag{2.6}$$

where

$$H(\varphi)=\begin{pmatrix}\varepsilon u-p\\ \varepsilon v-q\\ -\varepsilon p+\beta(-\Delta)p+\varepsilon^2u+(1-\beta\varepsilon)(-\Delta)u\\ -\varepsilon q+\beta(-\Delta)^m q+\varepsilon^2 v+(1-\beta\varepsilon)(-\Delta)^m v\end{pmatrix}, \tag{2.7}$$

$$F(\varphi)=\begin{pmatrix}0\\0\\ [1-M(\|\nabla u\|^2+\|\nabla^m v\|^2)](-\Delta)u-g_1(u,v)+f_1(x)\\ [1-M(\|\nabla u\|^2+\|\nabla^m v\|^2)](-\Delta)^m v-g_2(u,v)+f_2(x)\end{pmatrix}, \tag{2.8}$$

Lemma2.1.1. For $\forall\varphi=(u,v,p,q)^T\in E_0$,we get

$$(H(\varphi),\varphi)_{E_0}\ge\frac{\varepsilon}{4}\|\varphi\|_{E_0}^2+\frac{\beta}{2}(\|\nabla p\|^2+\|\nabla^m q\|^2). \tag{2.9}$$

Proof. By (2.4),(2.7) we get

$$\begin{aligned}(H(\varphi),\varphi)_{E_0}&=(\varepsilon\nabla u-\nabla p,\nabla u)+(-\varepsilon p+\beta(-\Delta)p+\varepsilon^2u+(1-\beta\varepsilon)(-\Delta)u,p)\\&\quad+(\varepsilon\nabla^m v-\nabla^m q,\nabla^m v)+(-\varepsilon q+\beta(-\Delta)^m q+\varepsilon^2 v+(1-\beta\varepsilon)(-\Delta)^m v,q)\\&=\varepsilon\|\nabla u\|^2-\varepsilon\|p\|^2+\beta\|\nabla p\|^2+\varepsilon^2(u,p)-\beta\varepsilon(\nabla u,\nabla p)+\\&\quad\varepsilon\|\nabla^m v\|^2-\varepsilon\|q\|^2+\beta\|\nabla^m q\|^2+\varepsilon^2(v,q)-\beta\varepsilon(\nabla^m v,\nabla^m q).\end{aligned} \tag{2.10}$$

By employing holder's inequality, Young's inequality and Poincare inequality, we process the terms in (2.10), we have

$$\varepsilon^2(u,p)\ge-\frac{\varepsilon^2}{2}\|u\|^2-\frac{\varepsilon^2}{2}\|p\|^2\ge-\frac{\varepsilon^2}{2\lambda_1}\|\nabla u\|^2-\frac{\varepsilon^2}{2}\|p\|^2. \tag{2.11}$$

$$\varepsilon^2(v,q)\ge-\frac{\varepsilon^2}{2}\|v\|^2-\frac{\varepsilon^2}{2}\|q\|^2\ge-\frac{\varepsilon^2}{2\lambda_1^m}\|\nabla^m v\|^2-\frac{\varepsilon^2}{2}\|q\|^2. \tag{2.12}$$

$$-\beta\varepsilon(\nabla u,\nabla p)\ge-\frac{\beta\varepsilon}{2}\|\nabla u\|^2-\frac{\beta\varepsilon}{2}\|\nabla p\|^2. \tag{2.13}$$

$$-\beta\varepsilon(\nabla^m v,\nabla^m q)\ge-\frac{\beta\varepsilon}{2}\|\nabla^m v\|^2-\frac{\beta\varepsilon}{2}\|\nabla^m q\|^2. \tag{2.14}$$

By the value of ε ,and substituting (2.11)-(2.14) ,we have

$$(H(\varphi),\varphi)_{E_0} \ge (\varepsilon - \frac{\beta\varepsilon}{2} - \frac{\varepsilon^2}{2\lambda_1})\|\nabla u\|^2 + (\frac{\beta}{2} - \frac{\beta\varepsilon}{2})\|\nabla p\|^2 + (-\frac{\varepsilon^2}{2} - \varepsilon)\|p\|^2 + \frac{\beta}{2}\|\nabla p\|^2$$
$$+ (\varepsilon - \frac{\beta\varepsilon}{2} - \frac{\varepsilon^2}{2\lambda_1^m})\|\nabla^m v\|^2 + (\frac{\beta}{2} - \frac{\beta\varepsilon}{2})\|\nabla^m q\|^2 + (-\frac{\varepsilon^2}{2} - \varepsilon)\|q\|^2 + \frac{\beta}{2}\|\nabla^m q\|^2$$
$$\ge \frac{\varepsilon}{4}(\|\nabla u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2) + \frac{\beta}{2}(\|\nabla p\|^2 + \|\nabla^m q\|^2)$$
$$= \frac{\varepsilon}{4}\|\varphi\|_{E_0}^2 + \frac{\beta}{2}(\|\nabla p\|^2 + \|\nabla^m q\|^2)$$
.(2.15)

The proof is completed.

The linearized equations of (1.1)- (1.5), we have

$$U_{tt} + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(-\Delta)U + 2M'(\|\nabla u\|^2 + \|\nabla^m v\|^2)[(\nabla u,\nabla U) + (\nabla^m v,\nabla^m V)](-\Delta)u$$
$$+\beta(-\Delta)U_t + g_{1u}(u,v)U + g_{1v}(u,v)V = 0 ,$$
(2.16)

$$V_{tt} + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(-\Delta)^m V + 2M'(\|\nabla u\|^2 + \|\nabla^m v\|^2)[(\nabla u,\nabla U) + (\nabla^m v,\nabla^m V)](-\Delta)^m v$$
$$+\beta(-\Delta)^m V_t + g_{2u}(u,v)U + g_{2v}(u,v)V = 0 ,$$
(2.17)

$$U(x,t)\big|_{x\in\partial\Omega} = 0, V(x,t)\big|_{x\in\partial\Omega} = 0, \quad t>0, \qquad (2.18)$$
$$U(x,0) = \xi_1, U_t(x,0) = \eta_1, V(x,0) = \xi_2, V_t(x,0) = \eta_2. \qquad (2.19)$$

where $(\xi_1,\xi_2,\eta_1,\eta_2)\in E_0, (u,v,p,q) = S(t)(u_0,v_0,p_0,q_0)$ is the solutions of (1.1)-(1.5) with $(u_0,v_0,p_0,q_0)\in A$. Given $(u_0,v_0,p_0,q_0)\in A$ and $S(t):E_0\to E_0$, the solution $S(t)(u_0,v_0,p_0,q_0)\in E_0$,

by stand methods we can show that for $\forall(\xi_1,\xi_2,\eta_1,\eta_2)\in E_0$,the linear initial boundary value problem(2.16)-(2.19) possess a unique solution $(U(t),V(t),P(t),Q(t))\in L^\infty((0,+\infty);E_0)$.

Theorem2.1.1. For $\forall t>0, R>0$,the mapping $S(t):E_0\to E_0$ is Frechet differentiable on E_0 .It's differential at $\varphi=(u_0,v_0,u_1,v_1)^T$ is the linear operator on $E_0:(\xi_1,\xi_2,\eta_1,\eta_2)^T\to (U(t),V(t),P(t),Q(t))^T$

,where $U(t),V(t)$ is the solution of(2.16)-(2.19).

Proof. let $\varphi_0=(u_0,v_0,p_0,q_0)^T\in E_0, \tilde{\varphi}_0=(u_0+\xi_1,v_0+\xi_2,p_0+\eta_1,q_0+\eta_2)^T\in E_0$ with $\|\varphi_0\|_{E_0}\le R$,

$\|\tilde{\varphi}_0\|_{E_0}\le R$,we denote $(u,v,p,q)^T=S(t)\varphi_0, (\tilde{u},\tilde{v},\tilde{p},\tilde{q})=S(t)\tilde{\varphi}_0$. We can get the Lipchitz property of $S(t)$ on the bounded set of E_0 ,that is

$$\|S(t)\varphi_0 - S(t)\tilde{\varphi}_0\|_{E_0}^2 \le e^{c_{19}t}\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^2. \qquad (2.20)$$

Let $\theta_1=\tilde{u}-u-U, \theta_2=\tilde{v}-v-V$, is the solution of problem:

$$\theta_{1tt} + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(-\Delta)\theta_1 + \beta(-\Delta)\theta_{1t} = h_1, \qquad (2.21)$$
$$\theta_{2tt} + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(-\Delta)^m\theta_2 + \beta(-\Delta)^m\theta_{2t} = h_2, \qquad (2.22)$$
$$\theta_1(0) = \theta_{1t}(0) = 0, \qquad (2.23)$$

$$\theta_2(0)=\theta_{2t}(0)=0, \qquad (2.24)$$

Setting $s=\|\nabla u\|^2+\|\nabla^m v\|^2$, $\tilde{s}=\|\nabla \tilde{u}\|^2+\|\nabla^m \tilde{v}\|^2$, then

$$\begin{aligned} h_1 &= (M(s)-M(\tilde{s}))(-\Delta)\tilde{u}+2M'(s)[(\nabla u,\nabla U)+(\nabla^m v,\nabla^m V)](-\Delta)u \\ &\quad + g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)U+g_{1v}(u,v)V \end{aligned} .$$

(2.25)

$$\begin{aligned} h_2 &= (M(s)-M(\tilde{s}))(-\Delta)^m\tilde{v}+2M'(s)[(\nabla u,\nabla U)+(\nabla^m v,\nabla^m V)](-\Delta)^m v \\ &\quad + g_2(u,v)-g_2(\tilde{u},\tilde{v})+g_{2u}(u,v)U+g_{2v}(u,v)V \end{aligned} .$$

(2.26)

Taking the scalar product of each side of (2.21) with θ_{1t},(2.22)with θ_{2t} ,and adding them together,

$$\frac{1}{2}\frac{d}{dt}(\|\theta_{1t}\|^2+\|\theta_{2t}\|^2)+M(s)\frac{1}{2}\frac{d}{dt}(\|\nabla\theta_1\|^2\,\|\nabla^m\theta_2\|^2)+\beta(\|\nabla\theta_{1t}\|^2+\|\nabla^m\theta_{2t}\|^2)=(h_1,\theta_{1t})+(h_2,\theta_{2t}).$$

(2.27)

Setting $\bar{u}=u-\tilde{u}$, $\bar{v}=v-\tilde{v}$, by (H_7), the mean value theorem and Lemma2.5.,we have

$$\begin{aligned} &((M(s)-M(\tilde{s}))(-\Delta)\tilde{u},\theta_{1t}) \\ &=(M'[\alpha s+(1-\alpha)\tilde{s}](\|\nabla u\|^2-\|\nabla\tilde{u}\|^2+\|\nabla^m v\|^2-\|\nabla^m\tilde{v}\|^2)(-\Delta)\tilde{u},\theta_{1t}) \qquad , \qquad (2.28) \\ &=(M'[\alpha s+(1-\alpha)\tilde{s}][(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})](-\Delta)\tilde{u},\theta_{1t}) \end{aligned}$$

$$\begin{aligned} &(\{M'[\alpha s+(1-\alpha)\tilde{s}]-M'(s)\}[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})](-\Delta)\tilde{u},\theta_{1t}) \\ &=M''(s_1)[\alpha s+(1-\alpha)\tilde{s}-s][(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})](-\Delta)\tilde{u},\theta_{1t}) \\ &=M''(s_1)(1-\alpha)(\tilde{s}-s)[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})](-\Delta)\tilde{u},\theta_{1t}) \qquad , \\ &\le c_1(\|\nabla\bar{u}\|+\|\nabla^m\bar{v}\|)(\|\nabla\bar{u}\|+\|\nabla^m\bar{v}\|)\|\nabla\theta_{1t}\| \\ &\le c_2(\|\nabla\bar{u}\|^2+\|\nabla^m\bar{v}\|^2)\|\nabla\theta_{1t}\| \end{aligned}$$

(2.29)

$$\begin{aligned} &(M'(s)[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})][(-\Delta)\tilde{u}-(-\Delta)u],\theta_{1t}) \\ &=(M'(s)[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})]\nabla\bar{u},\nabla\theta_{1t}) \\ &\le c_3(\|\nabla\bar{u}\|+\|\nabla^m\bar{v}\|)\|\nabla\bar{u}\|\|\nabla\theta_{1t}\| \qquad , \qquad (2.30) \\ &\le c_4(\|\nabla\bar{u}\|^2+\|\nabla^m\bar{v}\|^2)\|\nabla\theta_{1t}\| \end{aligned}$$

$$\begin{aligned} &(M'(s)[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})](-\Delta)u-2M'(s)[(\nabla u,\nabla\bar{u})+(\nabla^m v,\nabla^m\bar{v})](-\Delta)u,\theta_{1t}) \\ &=(M'(s)[(\nabla\bar{u},\nabla u+\nabla\tilde{u})+(\nabla^m\bar{v},\nabla^m v+\nabla^m\tilde{v})-2(\nabla u,\nabla\bar{u})-2(\nabla^m v,\nabla^m\bar{v})](-\Delta)u,\theta_{1t}) \\ &\le c_5(\|\nabla\bar{u}\|^2+\|\nabla^m\bar{v}\|^2)\|\nabla\theta_{1t}\| \end{aligned}$$

.(2.31)

Therefore, integrate (2.28)-(2.31), we have

$$\begin{aligned}&((M(s)-M(\tilde{s}))(-\Delta)\tilde{u}+2M'(s)[(\nabla u,\nabla U)+(\nabla^m v,\nabla^m V)](-\Delta)u,\theta_{1t})\\&\le c_6(\|\nabla\bar{\bar{u}}\|^2+\|\nabla^m\bar{\bar{v}}\|^2)\|\nabla\theta_{1t}\|\end{aligned}. \tag{2.32}$$

And similarly,

$$\begin{aligned}&((M(s)-M(\tilde{s}))(-\Delta)^m\tilde{v}+2M'(s)[(\nabla u,\nabla U)+(\nabla^m v,\nabla^m V)](-\Delta)^m v,\theta_{2t})\\&\le c_7(\|\nabla\bar{\bar{u}}\|^2+\|\nabla^m\bar{\bar{v}}\|^2)\|\nabla^m\theta_{2t}\|\end{aligned}.$$

(2.33)

By (H_4), (H_8) Lemma2.5.,thanks to

$$\begin{aligned}&(g_1(u,v)-g_1(u,v)-g_{1u}(u,v)\bar{\bar{u}},\theta_{1t})\\&\le\|g_1(u,v)-g_1(u,v)-g_{1u}(u,v)\bar{\bar{u}}\|\|\theta_{1t}\|\\&=\int_0^1\|[g_{1u}(\alpha u+(1-\alpha)u,v)-g_{1u}(u,v)]\bar{\bar{u}}\|\|\theta_{1t}\|,\\&\le c_8(\|\nabla\bar{\bar{u}}\|^{\delta_1+1}+\|\nabla^m\bar{\bar{v}}\|^{\delta_1}\|\nabla\bar{\bar{u}}\|)\|\theta_{1t}\|\end{aligned} \tag{2.34}$$

$$\begin{aligned}&(g_1(u,v)-g_1(u,\tilde{v})-g_{1v}(u,v)\bar{\bar{v}},\theta_{1t})\\&\le\|g_1(u,v)-g_1(u,\tilde{v})-g_{1v}(u,v)\bar{\bar{v}}\|\|\theta_{1t}\|\\&=\int_0^1\|[g_{1v}(u,\alpha\tilde{v}+(1-\alpha)v)-g_{1v}(u,v)]\bar{\bar{v}}\|\|\theta_{1t}\|.\\&\le c_9(\|\nabla\bar{\bar{u}}\|^{\delta_1}\|\nabla^m\bar{\bar{v}}\|+\|\nabla^m\bar{\bar{v}}\|^{\delta_1+1})\|\theta_{1t}\|\end{aligned} \tag{2.35}$$

Hence, by Young's inequality, we obtain

$$\begin{aligned}&(g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)U+g_{1v}(u,v)V,\theta_{1t})\\&=(g_1(u,v)-g_1(\tilde{u},v)+g_1(\tilde{u},v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)U+g_{1v}(u,v)V,\theta_{1t})\\&=(g_1(u,v)-g_1(\tilde{u},v)-g_{1u}(u,v)\bar{\bar{u}},\theta_{1t})+(g_1(\tilde{u},v)-g_1(\tilde{u},\tilde{v})-g_{1v}(u,v)\bar{\bar{v}},\theta_{1t})\\&\quad-(g_{1u}(u,v)\theta_1,\theta_{1t})-(g_{1v}(u,v)\theta_2,\theta_{1t})\\&\le(c_{10}\|\nabla\bar{\bar{u}}\|^{\delta_1+1}+c_{11}\|\nabla\bar{\bar{v}}\|^{\delta_1+1})\|\theta_{1t}\|+\|g_{1u}(u,v)\|_{L^\infty}\|\theta_1\|\|\theta_{1t}\|+\|g_{1v}(u,v)\|_{L^\infty}\|\theta_2\|\|\theta_{1t}\|\end{aligned}. \tag{2.36}$$

In the similar way,

$$\begin{aligned}&(g_2(u,v)-g_2(\tilde{u},\tilde{v})+g_{2u}(u,v)U+g_{2v}(u,v)V,\theta_{2t})\\&\le(c_{12}\|\nabla\bar{\bar{u}}\|^{\delta_1+1}+c_{13}\|\nabla\bar{\bar{v}}\|^{\delta_1+1})\|\theta_{2t}\|+\|g_{2u}(u,v)\|_{L^\infty}\|\theta_1\|\|\theta_{2t}\|+\|g_{2v}(u,v)\|_{L^\infty}\|\theta_2\|\|\theta_{2t}\|\end{aligned}.$$

(2.37)

By (H_4), (H_7) ,(2.32)-(2.33),(2.36)-(2.37),and Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[\|\theta_{1t}\|^2+\|\theta_{2t}\|^2+\mu(\|\nabla\theta_1\|^2+\|\nabla^m\theta_2\|^2)]\le c_{14}[\|\theta_{1t}\|^2+\|\theta_{2t}\|^2+\mu(\|\nabla\theta_1\|^2+\|\nabla^m\theta_2\|^2)]\\&\qquad+c_{15}(\|\nabla\bar{\bar{u}}\|^4+\|\nabla^m\bar{\bar{v}}\|^4+\|\nabla\bar{\bar{u}}\|^{\delta_1+1}+\|\nabla\bar{\bar{v}}\|^{\delta_1+1})\end{aligned}.$$

(2.38)

By Gronwall's inequality and (2.20), we obtain

$$\|\theta_{1t}\|^2+\|\theta_{2t}\|^2+\mu(\|\nabla\theta_1\|^2+\|\nabla^m\theta_2\|^2)\le c_{15}e^{c_{14}t}\int_0^t(\|\nabla\bar{\bar{u}}\|^4+\|\nabla^m\bar{\bar{v}}\|^4+\|\nabla\bar{\bar{u}}\|^{\delta_1+1}+\|\nabla\bar{\bar{v}}\|^{\delta_1+1})d\tau$$
$$\le c_{16}e^{c_{17}t}(\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^4+\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^{\delta_1+1}).$$

(2.39)

Since,

$$\frac{\|\tilde{\varphi}(t)-\varphi(t)-U(t)\|_{E_0}^2}{\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^2}\le c_{16}e^{c_{17}t}(\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^2+\|(\xi_1,\xi_2,\eta_1,\eta_2)\|_{E_0}^{\delta_1-1})\to 0,$$

(2.40)

as $(\xi_1,\xi_2,\eta_1,\eta_2)^T\to 0$ in E_0.The proof is completed.

2.2The Hausdorff Dimensions and Fractal Dimensions for the Global Attractor.

Consider the first varistion of (2.6) with initial condition:

$$\Psi_t{}'+P(\varphi)\Psi=\Gamma_1(\varphi)\Psi+\Gamma_2(\varphi)\Psi,\quad \Psi(0)=(\xi_1,\xi_2,\eta_1,\eta_2)^T\in E_0,t>0,$$

(2.41)

Where $\Psi=(U,V,P,Q)^T\in E_0,P=U_t+\varepsilon U,Q=V_t+\varepsilon V$ and $\varphi=(u,v,p,q)^T\in E_0$ is a solution of (2.6).

$$P(\varphi)=\begin{pmatrix}\varepsilon I & 0 & -I & 0\\ 0 & \varepsilon I & 0 & -I\\ \varepsilon^2 I+(1-\beta\varepsilon)(-\Delta) & 0 & -\varepsilon I+\beta(-\Delta) & 0\\ 0 & \varepsilon^2 I+(1-\beta\varepsilon)(-\Delta)^m & 0 & -\varepsilon I+\beta(-\Delta)^m\end{pmatrix},$$

(2.42)

$$\Gamma_1(\varphi)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-g_{1u}(u,v)&-g_{1v}(u,v)&0&0\\-g_{2u}(u,v)&-g_{2v}(u,v)&0&0\end{pmatrix},\qquad(2.43)$$

$$\Gamma_2(\varphi)=\begin{pmatrix}0\\0\\ [1-\mathrm{M}(\|\nabla u\|^2+\|\nabla^m v\|^2)](-\Delta)\mathrm{U}-2\mathrm{M}'(\|\nabla u\|^2+\|\nabla^m v\|^2)[(\nabla \mathrm{u},\nabla \mathrm{U})+(\nabla^m \mathrm{v},\nabla^m \mathrm{V})](-\Delta)\mathrm{u}\\ [1-\mathrm{M}(\|\nabla u\|^2+\|\nabla^m v\|^2)](-\Delta)^m\mathrm{V}-2\mathrm{M}'(\|\nabla u\|^2+\|\nabla^m v\|^2)[(\nabla \mathrm{u},\nabla \mathrm{U})+(\nabla^m \mathrm{v},\nabla^m \mathrm{V})](-\Delta)^m\mathrm{v}\end{pmatrix}$$

.(2.44)

It's easy to prove from Lemma2.1.2 that (2.41)is a well-posed problem in E_0,the mapping $S_\varepsilon(t):\{u_0,v_0,p_0=u_1+\varepsilon u_0,q_0=v_1+\varepsilon v_0\}\to\{u(t),v(t),p(t),q(t)\}$, $\varphi(t)=\{u(t),v(t),p(t),q(t)\}$ is

Frechet differentiable on E_0 for $\forall t\ge 0$,its differential at $\varphi=\{u_0,v_0,p_0,q_0\}^T$ is the linear operator on E_0, $(\xi_1,\xi_2,\eta_1,\eta_2)^T\to(U(t),V(t),P(t),Q(t))^T$,where $(U(t),V(t),P(t),Q(t))^T$ is the solution of (2.41).

Lemma2.2.1.[6] For any orthonormal family of elements of $(E_0,\|\cdot\|_{E_0})$, $(\xi_{1j},\xi_{2j},\eta_{1j},\eta_{2j})^T$, $j=1,2,\cdots,n_1$, we have

$$\sum_{j=1}^{n_1}(\left\|\nabla^{v}\xi_{1j}\right\|^{2}+\left\|\nabla^{mv}\xi_{2j}\right\|^{2})\le 2\sum_{j=1}^{n_1}(\lambda_j^{v-1}+\lambda_j^{mv-1}),\quad v\in[0,\frac{1}{m}).$$

(2.45)

where $\{\lambda_j\}_{j=1}^{+\infty}$ is the eigenvalue of $(-\Delta)$.

Proof. This is a direct consequence of Lemma VI 6.3 of [6].

Theorem2.2.1. If $(H_1)-(H_4),(H_6)$ hold, $\lambda_j,\eta_{1j},\eta_{2j}$ satisfying

$$-\frac{\beta}{4}\lambda_j\left\|\eta_{1j}\right\|^2+\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0\le\frac{\varepsilon}{16},$$

$-\frac{\beta}{4}\lambda_j^m\left\|\eta_{2j}\right\|^2+\frac{\lambda_j^{\frac{m}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{m}{2}}R_0^2k_0\le\frac{\varepsilon}{16}$,then there exists $\beta>0$,such that the Hausdorff and Fractalof dimensions of the global attractor A in E_0 satisfies

$$d_H(\mathrm{A})\le\min\left\{n_1\,\middle|\,n_1\in N,\frac{1}{n_1}\sum_{j=1}^{n_1}(\lambda_j^{v-1}+\lambda_j^{mv-1})<\frac{\varepsilon}{16c_{35}}\right\},\qquad(2.46)$$

$$d_F(\mathrm{A})\le 2\min\left\{n_1\,\middle|\,n_1\in N,\frac{1}{n_1}\sum_{j=1}^{n_1}(\lambda_j^{v-1}+\lambda_j^{mv-1})<\frac{\varepsilon}{16c_{35}}\right\},\qquad(2.47)$$

where R_0 is as in Lemma2.5.,and

$$\delta=\begin{cases}\frac{(n-2)(p-1)-2}{2},\frac{n}{n-2}\le p<\frac{n+2}{n-2},n\ge 2,\\ 0,n<2\quad or\quad 0\le p\le\frac{n}{n-2},n\ge 2.\end{cases}\qquad(2.48)$$

Proof. Let $n_1\in N$ be fixed. Respect to m_1 solutions $\Psi_1,\Psi_2,\cdots,\Psi_{n_1}$ of (2.41).At a given time $\tau,s\ge\tau$,let $Q_{n_1}(s)$ denote the orthogonal projection in E_0 onto span $\{\Psi_1(s),\Psi_2(s),\cdots,\Psi_{n_1}(s)\}$ .Let $y_j(s)=(\xi_1,\xi_2,\eta_1,\eta_2)^T\in E_0,\ j=1,2,\mathrm{L}\ ,n_1$ be an orthonormal basis of

$$Q_{n_1}(s)E_0=span\{\Psi_1(s),\Psi_2(s),\cdots,\Psi_{n_1}(s)\},\qquad(2.49)$$

With consider the inner product $(,)_{E_0}$ and norm $\|\cdot\|_{E_0}$.

Assume

$$\varphi(s)=(u(s),v(s),p(s),q(s))^T\in\mathrm{A}.\qquad(2.50)$$

Then $\|\varphi(s)\|_{E_0}\le M_0,\forall s\ge\tau$ .By $\|y_j\|_{E_0}=1$ and Lemma2.1.1.,we have

$$-(P(\varphi(s))y_j(s),y_j(s))_{E_0}\le-\frac{\varepsilon}{4}-\frac{\beta}{2}\left\|\nabla\eta_{1j}\right\|^2-\frac{\beta}{2}\left\|\nabla^m\eta_{2j}\right\|^2.$$

(2.51)

By (H_4)

$$\begin{aligned}&(\Gamma_1(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le(g_{1u}(u,v)\xi_{1j},\eta_{1j})+(g_{1v}(u,v)\xi_{2j},\eta_{1j})+(g_{2u}(u,v)\xi_{1j},\eta_{2j})+(g_{2v}(u,v)\xi_{2j},\eta_{2j})\\ &\le c_{29}\left\|\nabla^{-1}\xi_{1j}\right\|\left\|\nabla\eta_{1j}\right\|+c_{30}\left\|\nabla^{-1}\xi_{2j}\right\|\left\|\nabla\eta_{1j}\right\|+c_{31}\left\|\nabla^{-m}\xi_{1j}\right\|\left\|\nabla^m\eta_{2j}\right\|+c_{32}\left\|\nabla^{-m}\xi_{2j}\right\|\left\|\nabla^m\eta_{2j}\right\|\end{aligned}.$$

(2.52)

Then, by the Soblev embedding theorem:

$$H_0^{mv}(\Omega)\subset H^{mv}(\Omega)\subset H^{v}(\Omega)\subset H^{-v}(\Omega)\subset H^{-mv}(\Omega),\quad v\in[0,1]. \tag{2.53}$$

Therefore,

$$\begin{aligned}&(\Gamma_1(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le c_{33}\|\nabla^{\delta}\xi_{1j}\|\|\nabla\eta_{1j}\|+c_{34}\|\nabla^{m\delta}\xi_{2j}\|\|\nabla\eta_{1j}\|+c_{35}\|\nabla^{\delta}\xi_{1j}\|\|\nabla^{m}\eta_{2j}\|+c_{36}\|\nabla^{m\delta}\xi_{2j}\|\|\nabla^{m}\eta_{2j}\|.\\ &\le c_{37}(\|\nabla^{\delta}\xi_{1j}\|^2+\|\nabla^{m\delta}\xi_{2j}\|^2)+\frac{\beta}{4}\|\nabla\eta_{1j}\|^2+\frac{\beta}{4}\|\nabla^{m}\eta_{2j}\|^2\end{aligned}$$

(2.54)

By (H_6), (H_7) and Young's inequality, we have

$$\begin{aligned}&(\Gamma_2(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le(1-\mu_0)\|\nabla\xi_{1j}\|\|\nabla\eta_{1j}\|+2R_0^2k_0\|\nabla\xi_{1j}\|\|\nabla\eta_{1j}\|+2R_0^2k_0\|\nabla^{m}\xi_{2j}\|\|\nabla\eta_{1j}\|\\ &\quad+(1-\mu_0)\|\nabla^{m}\xi_{2j}\|\|\nabla^{m}\eta_{2j}\|+2R_0^2k_0\|\nabla\xi_{1j}\|\|\nabla^{m}\eta_{2j}\|+2R_0^2k_0\|\nabla^{m}\xi_{2j}\|\|\nabla^{m}\eta_{2j}\|. \qquad (2.55)\\ &\le\frac{\lambda_j^{\frac12}(1-\mu_0)}{2}+2\lambda_j^{\frac12}R_0^2k_0+\frac{\lambda_j^{\frac m2}(1-\mu_0)}{2}+2\lambda_j^{\frac m2}R_0^2k_0\end{aligned}$$

Choose $\lambda_j,\eta_{1j},\eta_{2j}$ satisfying

$$-\frac{\beta}{4}\lambda_j\|\eta_{1j}\|^2+\frac{\lambda_j^{\frac12}(1-\mu_0)}{2}+2\lambda_j^{\frac12}R_0^2k_0\le\frac{\varepsilon}{16},-\frac{\beta}{4}\lambda_j^{m}\|\eta_{2j}\|^2+\frac{\lambda_j^{\frac m2}(1-\mu_0)}{2}+2\lambda_j^{\frac m2}R_0^2k_0\le\frac{\varepsilon}{16}\cdot$$

(2.56)

We obtain

$$\begin{aligned}p_{n_1}(s)&=\sum_{j=1}^{n_1}((-P(\varphi(s))+\Gamma_1(\varphi(s))+\Gamma_2(\varphi(s)))y_j(s),y_j(s))_{E_0}\\ &\le\sum_{j=1}^{n_1}(-\frac{\varepsilon}{4}-\frac{\beta}{2}\|\nabla\eta_{1j}\|^2-\frac{\beta}{2}\|\nabla^{m}\eta_{2j}\|^2+\frac{\beta}{4}\|\nabla\eta_{1j}\|^2+\frac{\beta}{4}\|\nabla^{m}\eta_{2j}\|^2+\frac{\lambda_j^{\frac12}(1-\mu_0)}{2}+2\lambda_j^{\frac12}R_0^2k_0\\ &\quad+\frac{\lambda_j^{\frac m2}(1-\mu_0)}{2}+2\lambda_j^{\frac m2}R_0^2k_0)+\sum_{j=1}^{n_1}c_{35}(\|\nabla^{\delta}\xi_{1j}\|^2+\|\nabla^{m\delta}\xi_{2j}\|^2).\\ &\le-\frac{\varepsilon}{8}n_1+2c_{35}\sum_{j=1}^{n_1}(\lambda_j^{\delta-1}+\lambda_j^{m\delta-1})\end{aligned}$$

(2.57)

If $\frac{\varepsilon}{16c_{35}}\ge\frac{1}{n_1}\sum_{j=1}^{n_1}(\lambda_j^{\delta-1}+\lambda_j^{m\delta-1})$, then

$$q_{n_1}=\liminf_{t\to\infty}\sup_{\tau\in R}\sup_{\Phi\subset E_0}\sup_{\varphi(\tau)\in A}\frac{1}{t}\int_{\tau}^{\tau+t}p_{n_1}(s)ds\le-2n_1c_{35}(\frac{\varepsilon}{16c_{35}}-\frac{1}{n_1}\sum_{j=1}^{n_1}(\lambda_j^{\delta-1}+\lambda_j^{m\delta-1}))<0.$$

(2.58)

which imply that we obtain (2.46),(2.47) by Lemma 4 of [2].
The proof of Theorem 2.2.2. is completed.

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