THE GLOABAL ATTRACTOR FOR A CLASS OF HIGHER-ORDER COUPLED KIRCHHOFF-TYPE EQUATIONS WITH STRONG LINEAR DAMPING

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ABSTRACT

The paper investigates the longtime behavior of the initial boundary value problem for a system of coupled wave equations of higher-order Kirchhoff type with strong damping terms. Under the appropriate assumptions, we conclude the existence and uniqueness of solution by a priori estimate. After that, we show that the corresponding continuous solution semigroup possesses a global attractor which is compact by the method of operator semigroup.

Keywords: Higher-order Kirchhoff type; The existence and uniqueness; Priori estimates; Global attractor.
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1.INTRODUCTION

In this papar, we are concerned with the following coupled equations with nonlinear higher-order Kirchhoff type:

$$u_{tt}+M(\|D^m u\|^2+\|D^m v\|^2)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x),\quad \text{in } \Omega\times[0,+\infty),\tag{1.1}$$

$$v_{tt}+M(\|D^m u\|^2+\|D^m v\|^2)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x),\quad \text{in } \Omega\times[0,+\infty),\tag{1.2}$$

$$u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x),\qquad x\in\Omega,\tag{1.3}$$

$$v(x,0)=v_0(x),\qquad v_t(x,0)=v_1(x),\qquad x\in\Omega,\tag{1.4}$$

$$\frac{\partial^i u}{\partial n^i}=0,\qquad \frac{\partial^i v}{\partial n^i}=0,\qquad i=0,1,2,\ldots,m-1,\qquad x\in\partial\Omega,\ t\ge 0,\tag{1.5}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\beta>0$ is real number and $m\ge 1$ is positive integer. $M(s)$ is a nonnegative C^1 function. n denotes the unit outward normal vector on $\partial\Omega$, and $\frac{\partial^i}{\partial n^i}$ represents the i th order normal derivation. $g_j(u,v)$ and $f_j(x)(j=1,2)$ are given functions to be specified later.

To promote our study, let's take a look at some results concerning wave equations with Kirchhoff type. For the form of the single wave equation with Kirchhoff type :

$$u_{tt}+M(\|\nabla u\|_2^2)Au+g(u_t)=f(u),\tag{1.6}$$

in the case of $M\equiv 1$, it has been extensively studied and various results regarding existence and nonexistence of solutions have been demonstrated[1-5]. While as for M is not a constant function. S.T.Wu and L.Y.Tsai[6] proved (1.6) the blow-up of solutions for the strong dissipative term $-\Delta u_t$ and linear dissipative term u_t, and also showed the nonexistence of global solutions for the nonlinear dissipative term $|u_t|^{m-2}u_t$ with $m>2$. K Ono[7] constructed a unique global weak solution of (1.6) when $g(u_t) = \delta Au'$ and showed blowing up of a local solution with $M(s) = s^\gamma$ and $f(u) = |u|^\alpha u$ under the condition that initial energy is negative. In addition, the global existence of solution of quasi-linear wave equation was derived by many authors[8-9]. For the form of High-order Kirchhoff type. Fucai Li[10] studied the higher-order Kirchhoff type equation with nonlinear dissipation:

$$u_{tt} + \left(\int_\Omega |D^m u|^2 dx\right)^q (-\Delta)^m u + u_t |u_t|^r = |u|^p u, \quad \text{in } \Omega\times(0,+\infty), \tag{1.7}$$

he obtained that solution exists globally if $p \le r$, while if $p > \max\{r, 2p\}$, the solution with negative initial energy blows up at finite time.

Guoguang Lin, Yunlong Gao and Yuting Sun[11] considered the following higher-order Kirchhoff equation:

$$u_{tt} + (-\Delta)^m u_t + \|D^m u\|^{2q} (-\Delta)^m u + g(u) = f(x), \quad \text{in } \Omega\times(0,+\infty), \tag{1.8}$$

they acquired the existence and uniqueness of solution and the existence of global attractor, the estimation of the upper bounds of the Hausdorff and Fractal dimensions for the attractor were set up later. Guoguang Lin, Yunlong Gao[12] discussed the longtime behavior of solution for a system of strongly damped higher-order Kirchhoff type equation:

$$u_{tt} + (-\Delta)^m u_t + (\alpha + \beta\|D^m u\|^2)^q (-\Delta)^m u + g(u) = f(x), \quad \text{in } \Omega\times(0,+\infty), \tag{1.9}$$

they got the existence and uniqueness of the solution by the Galerkin method and obtained the existence of the global attractor in $H_0^m(\Omega)\times L^2(\Omega)$ according to the attractor theorem, besides, the estimation of the upper bound of Hausdorff dimension for the attractor was established.

Qingyong Gao, Fushan Li and Yanguo Wang[13] proved that the solution blows up in finite time under suitable conditions on the initial data,

$$u_{tt} + M(\|D^m u\|_2^2)(-\Delta)^m u + |u_t|^{q-2} u_t = |u|^p u, \quad \text{in } \Omega\times(0,+\infty). \tag{1.10}$$

As far as the system of equations with nonlinear Kirchhoff type, let us recall some results regarding it. J.J.Bae and S.S.Kim[14-15] consider the global existence and decay rates of solutions for the transmission problem of Kirchhoff type wave equations consisting of two physically different types of materials. S.T.Wu[16] considered the decay estimates of the energy function and the blow up of solutions in finite time when the initial energy is nonnegative,

$$\begin{aligned} &u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1} u = f_1(u,v), \\ &v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + \int_0^t h(t-s)\Delta u(s)ds + |v_t|^{p-1} v = f_2(u,v). \end{aligned} \tag{1.11}$$

Yaojun Ye[17] dealt with the existence of global solutions by constructing a stable set in $H_0^{m_1}(\Omega)\times H_0^{m_2}(\Omega)$ and gave the decay estimate of global solutions by applying a lemma of V.Komornik,

$$\begin{aligned} &u_{tt} + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(-\Delta)^{m_1} u + a|u_t|^{q-2} u_t = f_1(u,v), \\ &v_{tt} + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(-\Delta)^{m_1} v + a|v_t|^{q-2} v_t = f_2(u,v). \end{aligned} \tag{1.12}$$

Motivated on the above, our first purpose of the present paper is to show a unique global existence of solution for the problem (1.1)-(1.5); our second purpose is to obtain the existence of global attractor.

2.PRELIMINARIES AND MAIN RESULTS

In this section, to state our main results, we need the following assumptions and notations. We utilize the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their normal inner products and norms. Let $\| \ \|$ and $\| \ \|_p$ denote the normal $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, meanwhile we introduce $H^m(\Omega)=\left\{v\left|D^i v\in L^2(\Omega),|i|\le m\right.\right\}$, and

add the following abbreviations. Let $A=-\Delta$ with domain $D(A)=H^2(\Omega)\bigcap H_0^1(\Omega)$. We consider a family of Hilbert spaces $V_a=D(A^{a/2}), a\in R$, whose inner products and norms are given by $(\cdot,\cdot)_{V_a}=(A^{a/2},A^{a/2})$ and $\|\cdot\|_{V_a}=\|A^{a/2}\cdot\|$.

obviously,

$$V_0=L^2(\Omega),\ \ V_m=H^m\times H_0^1,\ \ V_{2m}=H^{2m}\times H_0^1,$$

let

$$E_0=V_m\times V_0\times V_m\times V_0,$$
$$E_1=V_{2m}\times V_m\times V_{2m}\times V_m.$$

We make the following hypotheses.

(H1) $M\in C^1([0.+\infty),R)$ is not decreasing and for positive constants m_0,m_1,

(1) $0<\beta<m_0\le M(s)$,

(2) $M(s)s\ge\int_0^s M(\tau)d\tau$, for all $s\ge 0$,

(3) $M'(s)\le M(s)$,

(H2) $g_i:R^{n+1}\to R(i=1,2)$ and $G_1(u,v)=\int_0^u g_1(\xi,v)d\xi$, $G_2(u,v)=\int_0^v g_2(u,\eta)d\eta$. For any $u,v\in V$, set $J(u,v)=\int_\Omega[G_1(u,v)+G_2(u,v)]$, there are $C\ge 0, C(\mu_1)\ge 0,\ C(\mu_2)\ge 0$, and for any $\mu_1\ge 0,\mu_2\ge 0$, we have

$$\left(g_1(u,v),u\right)+\left(g_2(u,v),v\right)-CJ(u,v)\ge-\mu_1(\|D^m u\|^2+\|D^m v\|^2)-C(\mu_1),$$
$$J(u,v)\ge-\mu_2(\|D^m u\|^2+\|D^m v\|^2)-C(\mu_2).$$

(H3) $g_i(i=1,2)$ are differentiable non-decreasing functions such that

$$\begin{cases}|g_1(u,v)|\le C_1(1+|u|^r+|v|^r),\\ |g_2(u,v)|\le C_2(1+|u|^{r'}+|v|^{r'}).\end{cases}$$

Lemma 2.1([18]) Let ψ be an absolutely continuous positive function on R^+, which satisfies for some $\varepsilon>0$ the differential inequality

$$\frac{d}{dt}\psi(t)+2\varepsilon\psi(t)\le g(t)\psi(t)+h(t),\quad t>0, \tag{2.1}$$

where $h\in L^1_{loc}(R^+)$ and

$$\int_\tau^t g(y)dy\le\varepsilon(t-\tau)+m,\quad for\ t\ge\tau\ge 0,$$

with some $m>0$. Then

$$\psi(t)\le e^m\left(\psi(s)e^{-\varepsilon(t-s)}+\int_s^t|h(y)|e^{-\varepsilon(t-y)}dy\right),\quad \forall t\ge s\ge 0. \tag{2.2}$$

Theorem 2.1 Let (u,p,v,q) be the solution of system (1.1)-(1.5). Assume(H1)-(H2), $f_1(x),\ f_2(x)\in L^2(\Omega)$, $(u_0,p_0,v_0,q_0)\in E_0$. Then it satisfies $(u,p,v,q)\in L^\infty\left((0,+\infty);E_0\right)$, and

$$\mathrm{E}_0(t)\le \frac{\varphi(0)}{k_3}e^{-\varepsilon_0 t}+\frac{1}{k_3}\left(\frac{2k_1}{\varepsilon_0}+\frac{C}{2{\varepsilon_0}^2}(\|f_1\|^2+\|f_2\|^2)\right), \tag{2.3}$$

where $p=u_t+\varepsilon u, q=v_t+\varepsilon v$, $E_0(t)=\|D^m u\|^2+\|p\|^2+\|D^m v\|^2+\|q\|^2$.

Thus there exists a positive constant $c(R_0)$ and $t_0=t_0(\Omega)$,

$$\|(u,p,v,q)\|_{E_0}^2=\|D^m u\|^2+\|p\|^2+\|D^m v\|^2+\|q\|^2\le c(R_0). \tag{2.4}$$

Proof. Multiplying equation (1.1) by $p=u_t+\varepsilon u$ and integrating it over Ω, we have

$$(u_{tt},p)=\left(p_t-\varepsilon(p-\varepsilon u),p\right)=\frac{1}{2}\frac{d}{dt}\|p\|^2-\varepsilon\|p\|^2+\varepsilon^2(u,p), \tag{2.5}$$

$$\begin{aligned}\left(M(\|D^m u\|^2+\|D^m v\|^2)(-\Delta)^m u,p\right)&=\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|D^m u\|^2\\&\quad+\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|D^m u\|^2,\end{aligned} \tag{2.6}$$

$$\left(\beta(-\Delta)^m u_t,p\right)=\beta\|D^m p\|^2-\beta\varepsilon(D^m u,D^m p), \tag{2.7}$$

$$\left(g_1(u,v),p\right)=\frac{d}{dt}\int_\Omega\int_0^u g_1(\xi,v)d\xi dx+\varepsilon\left(g_1(u,v),u\right), \tag{2.8}$$

then summing (2.5)-(2.8) up with other inner product terms

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|p\|^2+\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|D^m u\|^2+\frac{d}{dt}\int_\Omega\int_0^u g_1(\xi_0,v)d\xi dx\\&\quad+\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|D^m u\|^2-\varepsilon\|p\|^2+\varepsilon^2(u,p)+\beta\|D^m p\|^2\\&\quad-\beta\varepsilon(D^m u,D^m p)+\varepsilon\left(g_1(u,v),u\right)=\left(f_1(x),p\right),\end{aligned} \tag{2.9}$$

similarly, multiplying equation (1.2) by $q=v_t+\varepsilon v$ and integrating it over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|q\|^2+\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|D^m v\|^2+\frac{d}{dt}\int_\Omega\int_0^v g_2(u,\eta_0)d\eta dx\\&\quad+\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|D^m v\|^2-\varepsilon\|q\|^2+\varepsilon^2(v,q)+\beta\|D^m q\|^2\\&\quad-\beta\varepsilon(D^m v,D^m q)+\varepsilon\left(g_2(u,v),v\right)=\left(f_2(x),q\right).\end{aligned} \tag{2.10}$$

Taking $(2.9)+(2.10)$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\int_0^{\|D^m u\|^2+\|D^m v\|^2}M(\tau)d\tau+\|p\|^2+\|q\|^2+2J(u,v)\right]\\&\quad+\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u\|^2+\|D^m v\|^2)\\&\quad-\varepsilon(\|p\|^2+\|q\|^2)+\varepsilon^2\left((u,p)+(v,q)\right)+\beta(\|D^m p\|^2+\|D^m q\|^2)\\&\quad-\beta\varepsilon\left((D^m u,D^m p)+(D^m v,D^m q)\right)+\varepsilon\left(g_1(u,v),u\right)+\varepsilon\left(g_2(u,v),v\right)\\&\quad=\left(f_1(x),p\right)+\left(f_2(x),q\right).\end{aligned} \tag{2.11}$$

By applying the Holder inequality, Young's inequality and Poincare inequality, we deal with the several terms of (2.11) as follows

$$-\varepsilon(\|p\|^2+\|q\|^2)+\varepsilon^2\left((u,p)+(v,q)\right)=(-\varepsilon-\frac{\varepsilon^2}{2})(\|p\|^2+\|q\|^2)-\frac{\varepsilon^2}{2}(\|u\|^2+\|v\|^2), \quad (2.12)$$

$$-\frac{\varepsilon^2}{2}(\|u\|^2+\|v\|^2)\geq-\frac{\varepsilon^2}{2\lambda_1^m}(\|D^m u\|^2+\|D^m v\|^2), \quad (2.13)$$

$$\begin{aligned}&\beta(\|D^m p\|^2+\|D^m q\|^2)-\beta\varepsilon\left((D^m u,D^m p)+(D^m v,D^m q)\right)\\&\geq-\frac{\beta\varepsilon}{2}(\|D^m u\|^2+\|D^m v\|^2)+(\lambda_1^m\beta-\frac{\lambda_1^m\beta\varepsilon}{2})(\|p\|^2+\|q\|^2),\end{aligned} \quad (2.14)$$

with $0<\varepsilon<\min\left\{2,\frac{3\lambda_1^m(m_0-\beta)}{2},\frac{-(\lambda_1^m\beta+4)+\sqrt{(\lambda_1^m\beta+4)^2+8\lambda_1^m\beta}}{2}\right\}$.

Inserting (2.12)-(2.14) into (2.11), and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\int_0^{\|D^m u\|^2+\|D^m v\|^2}M(\tau)d\tau+\|p\|^2+\|q\|^2+2J(u,v)\right]\\&+\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u\|^2+\|D^m v\|^2)\\&+\left(\lambda_1^m\beta-(\frac{\lambda_1^m\beta}{2}+2)\varepsilon-\frac{\varepsilon^2}{2}\right)(\|p\|^2+\|q\|^2)\\&+(-\frac{\varepsilon^2}{2\lambda_1^m}-\frac{\beta\varepsilon}{2})(\|D^m u\|^2+\|D^m v\|^2)\\&\leq-\varepsilon\left(g_1(u,v),u\right)-\varepsilon\left(g_2(u,v),v\right)+\frac{1}{4\varepsilon}(\|f_1\|^2+\|f_2\|^2).\end{aligned} \quad (2.15)$$

By using assumption (H1), we have

$$\begin{aligned}&\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u\|^2+\|D^m v\|^2)\\&\geq\frac{\varepsilon}{4}\int_0^{\|D^m u\|^2+\|D^m v\|^2}M(\tau)d\tau+\frac{3m_0\varepsilon}{4}(\|D^m u\|^2+\|D^m v\|^2),\end{aligned} \quad (2.16)$$

and utilizing hypothesis (H2) with $\mu_1=\frac{\beta}{4}$ and $k_1=k_1(\varepsilon)=\varepsilon C(\frac{\beta}{4})$, we get

$$-\varepsilon\left(g_1(u,v),u\right)-\varepsilon\left(g_2(u,v),v\right)\leq-\varepsilon CJ(u,v)+\frac{\beta\varepsilon}{4}(\|D^m u\|^2+\|D^m v\|^2)+k_1, \quad (2.17)$$

combining (2.16)-(2.17) implies that

$$\begin{aligned}&\frac{d}{dt}\left[\int_0^{\|D^m u\|^2+\|D^m v\|^2}M(\tau)d\tau+\|p\|^2+\|q\|^2+2J(u,v)\right]\\&+\frac{\varepsilon}{2}M(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u\|^2+\|D^m v\|^2)\\&+\left(2\lambda_1^m\beta-(\lambda_1^m\beta+4)\varepsilon-\varepsilon^2\right)(\|p\|^2+\|q\|^2)\\&+2\varepsilon CJ(u,v)+\left(-\frac{\varepsilon^2}{\lambda_1^m}+\frac{3\varepsilon(m_0-\beta)}{2}\right)(\|D^m u\|^2+\|D^m v\|^2)\\&\leq 2k_1+\frac{1}{2\varepsilon}(\|f_1\|^2+\|f_2\|^2).\end{aligned} \quad (2.18)$$

By using (H2) with $\mu_2=\frac{\beta}{6}$, and obtain

$$\begin{aligned}&\frac{1}{6}\int_0^{\|D^m u\|^2+\|D^m v\|^2} M(\tau)d\tau + J(u,v) + C(\mu_2)\\ &\geq \frac{\beta}{6}(\|D^m u\|^2 + \|D^m v\|^2) + J(u,v) + C(\frac{\beta}{6}) \geq 0,\end{aligned} \tag{2.19}$$

setting

$$\Phi(t) = \int_0^{\|D^m u\|^2+\|D^m v\|^2} M(\tau)d\tau + \|p\|^2 + \|q\|^2 + 2J(u,v), \tag{2.20}$$

there exists constants $k_2 = 2C(\frac{\beta}{6})$ and $k_3 = \min\left\{\frac{2}{3}\beta, 1\right\}$, such that

$$\Phi(t) + k_2 \geq k_3 \mathrm{E}_0(t) \geq 0, \tag{2.21}$$

where $E_0(t) = \|D^m u\|^2 + \|p\|^2 + \|D^m v\|^2 + \|q\|^2$, due to $C < 1,\ \varepsilon < 2,\ \varepsilon_0 = C\varepsilon$,

$$\varepsilon_0 = \min\left\{\frac{\varepsilon}{2}, \left(2\lambda_1^m\beta + (\lambda_1^m\beta + 4)\varepsilon - \varepsilon^2\right)\right\}, \tag{2.22}$$

denoting $\varphi(t) = \Phi(t) + k_2$,

$$\frac{d}{dt}\varphi(t) + \varepsilon_0\varphi(t) \leq 2k_1 + \frac{1}{2\varepsilon}(\|f_1\|^2 + \|f_2\|^2), \tag{2.23}$$

$$\varphi(t) \leq \varphi(0)e^{-\varepsilon_0 t} + \frac{2k_1}{\varepsilon_0} + \frac{C}{2\varepsilon_0^{\ 2}}(\|f_1\|^2 + \|f_2\|^2), \tag{2.24}$$

where we use the Gronwall inequality, and hence

$$\mathrm{E}_0(t) \leq \frac{\varphi(0)}{k_3}e^{-\varepsilon_0 t} + \frac{1}{k_3}\left[\frac{2k_1}{\varepsilon_0} + \frac{C}{2\varepsilon_0^{\ 2}}(\|f_1\|^2 + \|f_2\|^2)\right], \tag{2.25}$$

then

$$\overline{\lim_{t\to\infty}}\|(u,p,v,q)\|_{E_0}^2 \leq \mathrm{E}_0(t) \leq \frac{1}{k_3}\left(\frac{2k_1}{\varepsilon_0} + \frac{C}{2\varepsilon_0^{\ 2}}(\|f_1\|^2 + \|f_2\|^2)\right), \tag{2.26}$$

that is , there exists a positive constant $c(R_0)$ and $t_0 = t_0(\Omega)$, such that for $t > t_0$,

$$\|(u,p,v,q)\|_{E_0}^2 = \|D^m u\|^2 + \|p\|^2 + \|D^m v\|^2 + \|q\|^2 \leq c(R_0). \tag{2.27}$$

The proof of theorem 2.1 is completed.

Theorem 2.2 Let (u,p,v,q) be the solution of system(1.1)-(1.5). Assume(H1)-(H3), $f_1(x), f_2(x) \in H^m(\Omega)$, $(u_0,p_0,v_0,q_0) \in E_1$, $|(f_1,u)| + |(f_2,v)| \leq 2C_0$. Then it satisfies $(u,p,v,q) \in L^\infty\left((0,+\infty);E_1\right)$, and

$$E_1(t) \leq \frac{e^l\psi(0)e^{-k_4 t}}{k_5} + \frac{C_5 e^l}{k_4 k_5}(1 - e^{-k_4 t}), \tag{2.28}$$

where $E_1(t) = \|\Delta^m u\|^2 + \|D^m p\|^2 + \|\Delta^m v\|^2 + \|D^m q\|^2$.

Thus there exists a positive constant $c(R_1)$ and $t_1 = t_1(\Omega)$,

$$\|(u,p,v,q)\|_{E_1}^2 = \|\Delta^m u\|^2 + \|D^m p\|^2 + \|\Delta^m v\|^2 + \|D^m q\|^2 \leq c(R_1). \tag{2.29}$$

Proof. Multiplying equation (1.1) by $(-\Delta)^m p$ and integrating it over Ω, we have

$$\left(u_{tt}, (-\Delta)^m p\right) = \frac{1}{2}\frac{d}{dt}\|D^m p\|^2 - \varepsilon\|D^m p\|^2 + \varepsilon^2(D^m u, D^m p), \tag{2.30}$$

$$\left(M(\|D^m u\|^2+\|D^m v\|^2)(-\Delta)^m u,(-\Delta)^m p\right)=\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|\Delta^m u\|^2 \\ +\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|\Delta^m u\|^2, \tag{2.31}$$

$$\left(\beta(-\Delta)^m u_t,(-\Delta)^m p\right)=\beta\|\Delta^m p\|^2-\beta\varepsilon\left((-\Delta)^m u,(-\Delta)^m p\right), \tag{2.32}$$

then adding (2.30)-(2.32) up with part scalar product terms

$$\frac{1}{2}\frac{d}{dt}\|D^m p\|^2+\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|\Delta^m u\|^2 \\ +\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|\Delta^m u\|^2-\varepsilon\|D^m p\|^2 \\ +\varepsilon^2(D^m u,D^m p)+\beta\|\Delta^m p\|^2-\beta\varepsilon\left((-\Delta)^m u,(-\Delta)^m p\right) \\ +\left(g_1(u,v),(-\Delta)^m p\right)=(D^m f_1,D^m p), \tag{2.33}$$

and similarly, multiplying equation (1.2) by $(-\Delta)^m q$ and integrating it over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|D^m q\|^2+\frac{1}{2}M(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}\|\Delta^m v\|^2 \\ +\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)\|\Delta^m v\|^2-\varepsilon\|D^m q\|^2 \\ +\varepsilon^2(D^m v,D^m q)+\beta\|\Delta^m q\|^2-\beta\varepsilon\left((-\Delta)^m v,(-\Delta)^m q)\right) \\ +\left(g_2(u,v),(-\Delta)^m q\right)=(D^m f_2,D^m q). \tag{2.34}$$

Taking $(2.33)+(2.34)$, it follows that

$$\frac{1}{2}\frac{d}{dt}\left[M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|^2\right] \\ +\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)-\varepsilon(\|D^m p\|^2+\|D^m q\|^2) \\ +\varepsilon^2\left((D^m u,D^m p)+(D^m v,D^m q)\right)+\beta(\|\Delta^m p\|^2+\|\Delta^m q\|^2) \\ -\beta\varepsilon\left[\left((-\Delta)^m u,(-\Delta)^m p\right)+\left((-\Delta)^m v,(-\Delta)^m q\right)\right] \\ +\left[\left(g_1(u,v),(-\Delta)^m p\right)+\left(g_2(u,v),(-\Delta)^m q\right)\right] \\ =(D^m f_1,D^m p)+(D^m f_2,D^m q) \\ +\frac{1}{2}(\|\Delta^m u\|^2+\|\Delta^m v\|^2)M'(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}(\|D^m u\|^2+\|D^m v\|^2). \tag{2.35}$$

By utilizing the Holder inequality, Young's inequality and Poincare inequality, we handle the certain terms of (2.35) as follows

$$-\varepsilon(\|D^m p\|^2+\|D^m q\|^2)+\varepsilon^2\left((D^m u,D^m p)+(D^m v,D^m q)\right) \\ =(-\varepsilon-\frac{\varepsilon^2}{2})(\|D^m p\|^2+\|D^m q\|^2)-\frac{\varepsilon^2}{2}(\|D^m u\|^2+\|D^m v\|^2), \tag{2.36}$$

$$-\frac{\varepsilon^2}{2}(\|D^m u\|^2+\|D^m v\|^2)\geq-\frac{\varepsilon^2}{2\lambda_1{}^m}(\|\Delta^m u\|^2+\|\Delta^m v\|^2, \tag{2.37}$$

$$\beta(\|\Delta^m p\|^2+\|\Delta^m q\|^2)-\beta\varepsilon\left[\left(((-\Delta)^m u,(-\Delta)^m p)\right)+\left(((-\Delta)^m v,(-\Delta)^m q)\right)\right] \\ \geq -\frac{\beta\varepsilon}{2}(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+(\beta-\frac{\beta\varepsilon}{2})(\|\Delta^m p\|^2+\|\Delta^m q\|^2), \tag{2.38}$$

with $0<\varepsilon<\min\left\{\frac{2\beta}{\beta+1},\frac{-(\lambda_1^m\beta+\lambda_1^m+4)^2+\sqrt{(\lambda_1^m\beta+\lambda_1^m+4)^2+8\lambda_1^m\beta}}{2}\right\}$.

Putting (2.36)-(2.38) into (2.35), and get

$$\frac{1}{2}\frac{d}{dt}\left[M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|^2\right] \\ +\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2) \\ +\left(-2\varepsilon-\frac{\varepsilon^2}{2}\right)(\|D^m p\|^2+\|D^m q\|^2)+(\beta-\frac{\beta\varepsilon}{2})(\|\Delta^m p\|^2+\|\Delta^m q\|^2) \\ +(-\frac{\varepsilon^2}{2\lambda_1^m}-\frac{\beta\varepsilon}{2})(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\left[\left(g_1(u,v),(-\Delta)^m p\right)\right. \\ \left.+\left(g_2(u,v),(-\Delta)^m q\right)\right] \\ =\frac{1}{4\varepsilon}(\|D^m f_1\|^2+\|D^m f_2\|^2)+\frac{1}{2}(\|\Delta^m u\|^2+\|\Delta^m v\|^2) \\ M'(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}(\|D^m u\|^2+\|D^m v\|^2). \tag{2.39}$$

By applying assumption (H1), we have

$$\varepsilon M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2) \\ \geq \frac{\varepsilon}{4}M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\frac{3m_0\varepsilon}{4}(\|\Delta^m u\|^2+\|\Delta^m v\|^2), \tag{2.40}$$

and using hypothesis (H3) , the Holder inequality, Young's inequality as well as Gagliardo-Nirenberg inequality, we get

$$\|g_1(u,v)\|^2=\int_\Omega |g_1(u,v)|^2dx\leq\int_\Omega \left|C_1(1+|u|^r+|v|^r)\right|^2dx \\ =C_1^2\int_\Omega(1+2|u|^r+2|v|^r+2|u|^{2r}+|v|^{2r}+2|u|^r|v|^r)dx \\ \leq C_1^2\int_\Omega(1+1+|u|^{2r}+1+|v|^{2r}+2|u|^{2r}+2|v|^{2r})dx \\ = 3C_1^2(1+\|u\|_{2r}^{2r}+\|v\|_{2r}^{2r}), \tag{2.41}$$

in a similar way

$$\|g_2(u,v)\|^2\leq 3C_2^2(1+\|u\|_{2r'}^{2r'}+\|v\|_{2r'}^{2r'}), \tag{2.42}$$

moreover , according to Gagliardo-Nirenberg inequality

$$\begin{cases}\|u\|_{2r}^{2r}\leq C\|D^m u\|^{\frac{n(r-1)}{m}}\|u\|^{\frac{2mr-n(r-1)}{m}}, \\ \|v\|_{2r}^{2r}\leq C\|D^m v\|^{\frac{n(r-1)}{m}}\|v\|^{\frac{2mr-n(r-1)}{m}},\end{cases}\quad \begin{cases}1\leq r\leq\frac{2n}{n-2m}, n>2m, \\ 1\leq r\leq\infty, \quad n\leq 2m.\end{cases}(or\ r') \tag{2.43}$$

consequently

$$\|g_1(u,v)\|^2+\|g_2(u,v)\|^2\leq C_3, \tag{2.44}$$

$$\left\|\left(g_1(u,v),(-\Delta)^m p\right)+\left(g_2(u,v),(-\Delta)^m q\right)\right\| \ge -\frac{C_3}{2\varepsilon}-\frac{\varepsilon}{2}(\|\Delta^m p\|^2+\|\Delta^m q\|^2). \tag{2.45}$$

Applying Poincare inequality to (2.38) and (2.45), we have

$$(\beta-\frac{\beta\varepsilon}{2}-\frac{\varepsilon}{2})(\|\Delta^m p\|^2+\|\Delta^m q\|^2)\ge \lambda_1^m(\beta-\frac{\beta\varepsilon}{2}-\frac{\varepsilon}{2})(\|D^m p\|^2+\|D^m q\|^2). \tag{2.46}$$

And utilizing Theorem 2.1 and Holder inequality, we get

$$\begin{aligned}
&\frac{1}{2}(\|\Delta^m u\|^2+\|\Delta^m v\|^2)M'(\|D^m u\|^2+\|D^m v\|^2)\frac{d}{dt}(\|D^m u\|^2+\|D^m v\|^2)\\
&\le (\|\Delta^m u\|^2+\|\Delta^m v\|^2)M'(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u\|^2+\|D^m v\|^2+\|D^m u_t\|^2+\|D^m v_t\|^2)\\
&\le C_4(\|\Delta^m u\|^2+\|\Delta^m v\|^2)M(\|D^m u\|^2+\|D^m v\|^2)(\|D^m u_t\|^2+\|D^m v_t\|^2) \qquad (2.47)\\
&\le C_4(\|D^m u_t\|^2+\|D^m v_t\|^2)\\
&\quad \left(M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|\right).
\end{aligned}$$

Substituting (2.40)-(2.47) into (2.39), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|^2\right]\\
&+\frac{\varepsilon}{2}M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)\\
&+\left(2\lambda_1^m\beta-(\lambda_1^m\beta+\lambda_1^m+4)\varepsilon-\varepsilon^2\right)(\|D^m p\|^2+\|D^m q\|^2)\\
&+\left(-\frac{\varepsilon^2}{\lambda_1^m}+\frac{\varepsilon(3m_0-2\beta)}{2}\right)(\|\Delta^m u\|^2+\|\Delta^m v\|^2) \qquad (2.48)\\
&\le \frac{1}{2\varepsilon}(\|D^m f_1\|^2+\|D^m f_2\|^2)+2C_4(\|D^m u_t\|^2+\|D^m v_t\|^2)\\
&\quad\left(M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|^2\right),
\end{aligned}$$

where we set

$$2k_4=\min\left\{\frac{\varepsilon}{2},2\lambda_1^m\beta-(\lambda_1^m\beta+\lambda_1^m+4)\varepsilon-\varepsilon^2\right\},$$

$$\psi(t)=M(\|D^m u\|^2+\|D^m v\|^2)(\|\Delta^m u\|^2+\|\Delta^m v\|^2)+\|D^m p\|^2+\|D^m q\|^2,$$

$$g(t)=2C_4(\|D^m u_t\|^2+\|D^m v_t\|^2), \tag{2.49}$$

and hence

$$\frac{d}{dt}\psi(t)+2k_4\psi(t)\le g(t)\psi(t)+C_5,\quad t>0. \tag{2.50}$$

Taking L^2-inner product by u_t in (1.1) and v_t in (1.2), combining them

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2+\|v_t\|^2+\int_0^{\|D^m u\|^2+\|D^m v\|^2}M(\tau)d\tau+2J(u,v)-2\left((f_1,u)+(f_2,v)\right)\right] \qquad (2.51)\\
&+\beta(\|D^m u_t\|^2+\|D^m v_t\|^2)=0
\end{aligned}$$

integrating the above formula over $[t,\infty)$, we get $\int_t^\infty 2C_4(\|D^m u_t\|^2+\|D^m v_t\|^2)<C_6$,

that is $$\int_\tau^t g(y)dy \le k_4(t-\tau)+l, \quad t\ge\tau\ge 0, \tag{2.52}$$
with some $l>0$.
Then
$$\psi(t)\le e^l\left(\psi(0)e^{-k_4 t}+C_5\int_0^t e^{-k_4(t-y)}dy\right), \quad \forall t\ge 0, \tag{2.53}$$
$$\psi(t)\le e^l\psi(0)e^{-k_4 t}+\frac{C_5 e^l}{k_4}(1-e^{-k_4 t}), \tag{2.54}$$
let $k_5=\min(\beta,1)$, and hence
$$E_1(t)\le\frac{e^l\psi(0)e^{-k_4 t}}{k_5}+\frac{C_5 e^l}{k_4 k_5}(1-e^{-k_4 t}). \tag{2.55}$$
And then
$$\overline{\lim_{t\to\infty}}\|(u,p,v,q)\|_{E_1}^2\le\frac{C_5 e^l}{k_4 k_5}, \tag{2.56}$$
that is , there exists a positive constant $c(R_1)$ and $t_1=t_1(\Omega)$, such that for $t>t_1$,
$$\|(u,p,v,q)\|_{E_1}^2=\|\Delta^m u\|^2+\|D^m p\|^2+\|\Delta^m v\|^2+\|D^m q\|^2\le c(R_1). \tag{2.57}$$
The proof of theorem 2.2 is completed.

3.GLOBAL ATTRACTOR

Theorem 3.1 Suppose that (H1)-(H3) hold, and $(u_0,p_0,v_0,q_0)\in E_1$, $f_1,f_2\in H^m(\Omega)$, then the initial boundary value problem (1.1)-(1.6) exists a unique smooth solution
$$\left(u(x,t),p(x,t),v(x,t),q(x,t)\right)\in L^\infty\left((0,+\infty);E_1\right). \tag{3.1}$$
Proof. Applying Theorem 2.1 and Theorem 2.2 with the Galerkin method, we can easily obtain the existence of solution. Next, let us prove the uniqueness of solution in detail.

Assume that $W_1=\begin{pmatrix}u_1\\v_1\end{pmatrix}$, $W_2=\begin{pmatrix}u_2\\v_2\end{pmatrix}$ are two different solutions of the problem (1.1)-(1.5), let $W=\begin{pmatrix}u\\v\end{pmatrix}=W_1-W_2$, then the two solutions satisfy
$$\begin{cases}u_{1tt}+M(\|D^m u_1\|^2+\|D^m v_1\|^2)(-\Delta)^m u_1+\beta(-\Delta)^m u_{1t}+g_1(u_1,v_1)=f_1(x),\\ v_{1tt}+M(\|D^m u_1\|^2+\|D^m v_1\|^2)(-\Delta)^m v_1+\beta(-\Delta)^m v_{1t}+g_2(u_1,v_1)=f_2(x),\\ u_1(x,0)=u_{01}(x);\quad u_{1t}(x,0)=u_{11}(x),\\ v_1(x,0)=v_{01}(x);\quad v_{1t}(x,0)=v_{11}(x).\end{cases} \tag{3.2}$$
$$\begin{cases}u_{2tt}+M(\|D^m u_2\|^2+\|D^m v_2\|^2)(-\Delta)^m u_2+\beta(-\Delta)^m u_{2t}+g_1(u_2,v_2)=f_1(x),\\ v_{2tt}+M(\|D^m u_2\|^2+\|D^m v_2\|^2)(-\Delta)^m v_2+\beta(-\Delta)^m v_{2t}+g_2(u_2,v_2)=f_2(x),\\ u_2(x,0)=u_{02}(x);\quad u_{2t}(x,0)=u_{12}(x),\\ v_2(x,0)=v_{02}(x);\quad v_{2t}(x,0)=v_{12}(x).\end{cases} \tag{3.3}$$
let (3.2) subtracts (3.3), we have

$$\begin{cases} u_{tt} + M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m u_1 - M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m u_2 \\ +\beta(-\Delta)^m u_t + g_1(u_1, v_1) - g_1(u_2, v_2) = 0, \\ v_{tt} + M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m v_1 - M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m v_2 \\ +\beta(-\Delta)^m v_t + g_2(u_1, v_1) - g_2(u_2, v_2) = 0. \end{cases} \tag{3.4}$$

Multiplying the first formula of (3.4) by u_t and integrating it over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \left(M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m u_1 - M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m u_2, u_t\right)$$
$$+\beta\|D^m u_t\|^2 + \left(g_1(u_1, v_1) - g_1(u_2, v_2), u_t\right) = 0, \tag{3.5}$$

multiplying the second formula of (3.4) by v_t and integrating it over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2 + \left(M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m v_1 - M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m v_2, v_t\right)$$
$$+\beta\|D^m v_t\|^2 + \left(g_2(u_1, v_1) - g_2(u_2, v_2), v_t\right) = 0, \tag{3.6}$$

taking (3.5) $+$ (3.6), it follows that

$$\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|v_t\|^2) + \left(M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m u_1\right.$$
$$\left. -M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m u_2, u_t\right) + \left(M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m v_1\right.$$
$$\left. -M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m v_2, v_t\right) + \beta(\|D^m u_t\|^2 + \|D^m v_t\|^2)$$
$$+\left(g_1(u_1, v_1) - g_1(u_2, v_2), u_t\right) + \left(g_2(u_1, v_1) - g_2(u_2, v_2), v_t\right) = 0. \tag{3.7}$$

By exploiting the Holder inequality, we obtain

$$\left(M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m (u_1 - u_2) + M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(-\Delta)^m u_2\right.$$
$$\left. -M(\|D^m u_2\|^2 + \|D^m v_2\|^2)(-\Delta)^m u_2, u_t\right)$$
$$= \frac{1}{2}M(\|D^m u_1\|^2 + \|D^m v_1\|^2)\frac{d}{dt}\|D^m u\|^2 + M'(\xi)\Big((\|D^m u_1\| + \|D^m u_2\|)(\|D^m u_1\| - \|D^m u_2\|)$$
$$+(\|D^m v_1\| + \|D^m v_2\|)(\|D^m v_1\| - \|D^m v_2\|)\Big)\|(-\Delta)^m u_2\|\|u_t\|,$$

(3.8)

and by applying Young's inequality and Theorem 2.1 and 2.2, we obtain

$$M'(\xi)\Big((\|D^m u_1\| + \|D^m u_2\|)(\|D^m u_1\| - \|D^m u_2\|)$$
$$+(\|D^m v_1\| + \|D^m v_2\|)(\|D^m v_1\| - \|D^m v_2\|)\Big)\|(\Delta)^m u_2\|\|u_t\|$$
$$\le C_7(\|D^m u\| + \|D^m v\|)\|u_t\|$$
$$\le C_7(\|D^m u\|^2 + \|D^m v\|^2 + \frac{\|u_t\|^2}{2}), \tag{3.9}$$

similarly

$$\Big(M(\|D^m u_1\|^2+\|D^m v_1\|^2)(-\Delta)^m(v_1-v_2)+M(\|D^m u_1\|^2+\|D^m v_1\|^2)(-\Delta)^m v_2$$
$$-M(\|D^m u_2\|^2+\|D^m v_2\|^2)(-\Delta)^m v_2, v_t\Big)$$
$$=\frac{1}{2}M(\|D^m u_1\|^2+\|D^m v_1\|^2)\frac{d}{dt}\|D^m v\|^2+M'(\eta)\Big((\|D^m u_1\|+\|D^m u_2\|)(\|D^m u_1\|-\|D^m u_2\|)$$
$$+(\|D^m v_1\|+\|D^m v_2\|)(\|D^m v_1\|-\|D^m v_2\|)\Big)\|(-\Delta)^m v_2\|\|v_t\|, \tag{3.10}$$

$$M'(\eta)\Big((\|D^m u_1\|+\|D^m u_2\|)(\|D^m u_1\|-\|D^m u_2\|)$$
$$+(\|D^m v_1\|+\|D^m v_2\|)(\|D^m v_1\|-\|D^m v_2\|)\Big)\|(-\Delta)^m v_2\|\|v_t\|$$
$$\le C_8(\|D^m u\|+\|D^m v\|)\|v_t\|$$
$$\le C_8(\|D^m u\|^2+\|D^m v\|^2+\frac{\|v_t\|^2}{2}), \tag{3.11}$$

by utilizing Young's inequality, Poincare inequality and Theorem 2.1 and 2.2, we get

$$|(g_1(u_1,v_1)-g_1(u_2,v_2),u_t)|$$
$$=|(g_1(u_1,v_1)-(g_1(u_1,v_2)+(g_1(u_1,v_2)-g_1(u_2,v_2),u_t)|$$
$$\le\left|(|v_1|^r-|v_2|^r, u_t)\right|+\left|(|u_1|^r-|u_2|^r, u_t)\right|$$
$$\le\left|C(1+|v_1|^{r-1}+|v_2|^{r-1})v, u_t)\right|+\left|C(1+|u_1|^{r-1}+|u_2|^{r-1})u, u_t)\right|$$
$$\le\left\|C(1+|v_1|^{r-1}+|v_2|^{r-1})\right\|_\infty\|v\|\|u_t\|+\left\|C(1+|u_1|^{r-1}+|u_2|^{r-1})\right\|_\infty\|u\|\|u_t\|$$
$$\le C_9(\|v\|^2+\|u\|^2+\frac{\|u_t\|^2}{2})$$
$$\le C_9(\lambda_1^{-m}\|D^m u\|^2+\lambda_1^{-m}\|D^m v\|^2+\frac{\|u_t\|^2}{2}), \tag{3.12}$$

in a similar way

$$|(g_2(u_1,v_1)-g_2(u_2,v_2),v_t)|\le C_{10}(\lambda_1^{-m}\|D^m u\|^2+\lambda_1^{-m}\|D^m v\|^2+\frac{\|v_t\|^2}{2}), \tag{3.13}$$

substituting (3.8)-(3.13) into (3.7), and Theorem 2.1 and 2.2, we have

$$\frac{d}{dt}\Big[M(\|D^m u_1\|^2+\|D^m v_1\|^2)(\|D^m u\|^2+\|D^m v\|^2)+\|u_t\|^2+\|v_t\|^2\Big]$$
$$\le\Big(2C_{11}+2C_{12}\lambda_1^{-m}+C_{13}M'(\|D^m u_1\|^2+\|D^m v_1\|^2)\Big)(\|D^m u\|^2+\|D^m v\|^2)$$
$$+(C_{11}+C_{12})(\|u_t\|^2+\|v_t\|^2), \tag{3.14}$$

setting $2C_{11}+2C_{12}\lambda_1^{-m}+C_{13}M'(\|D^m u_1\|^2+\|D^m v_1\|^2)\le C_{14}M(\|D^m u_1\|^2+\|D^m v_1\|^2)$,

and let $k_6=\max\{C_{14},C_{11}+C_{12}\}$, so

$$\frac{d}{dt}\Big(M(\|D^m u_1\|^2+\|D^m v_1\|^2)(\|D^m u\|^2+\|D^m v\|^2)+\|u_t\|^2+\|v_t\|^2\Big)$$
$$\le k_6\Big(M(\|D^m u_1\|^2+\|D^m v_1\|^2)(\|D^m u\|^2+\|D^m v\|^2)+\|u_t\|^2+\|v_t\|^2\Big), \tag{3.15}$$

where we set

$$\phi(t) = M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(\|D^m u\|^2 + \|D^m v\|^2) + \|u_t\|^2 + \|v_t\|^2 . \tag{3.16}$$

That is

$$\frac{d}{dt}\phi(t) \le k_6 \phi(t) , \tag{3.17}$$

then, a simple application of Gronwall inequality gives,

$$\phi(t) \le \phi(0) e^{k_6 t} , \tag{3.18}$$

thereby

$$M(\|D^m u_1\|^2 + \|D^m v_1\|^2)(\|D^m u\|^2 + \|D^m v\|^2) + \|u_t\|^2 + \|v_t\|^2 \le 0 . \tag{3.19}$$

That is

$$\|D^m u\| = \|D^m v\| = \|u_t\| = \|v_t\| = 0 , \tag{3.20}$$

namely

$$W = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{3.21}$$

From the above, the proof of theorem 3.1 is now completed.

Theorem 3.2([19]) Suppose that E be a Banach space, and $\{S(t)\}(t \ge 0)$ are the semigroup operator on E, $S(t): E \to E$, $S(t) \cdot S(\tau) = S(t+\tau)(\forall t, \tau \ge 0)$, $S(0) = I$, where I is a unit operator. Assume $S(t)$ satisfies the following conditions:

(1) $S(t)$ is uniformly bounded on E, namely, $\forall R > 0, \|u\|_E \le R$, there exists a constant $c(R)$, so that

$$\|S(t)u\|_E \le c(R) \ (\forall t \in [0, +\infty)) ;$$

(2) it exists a bounded absorbing set $B_0 \subset E$, namely, $\forall B \subset E$, there exists a constant $t_0 = t_0(B) > 0$, so that

$$S(t)B \subset B_0 \ (\forall t \ge t_0) ;$$

(3) $S(t)(t > 0)$ is complete continuity operator. Therefore, the semigroup has a compact global attractor A.

Theorem 3.3 Under the assumption of Theoerem 3.1, equations have global attractor

$$A = \omega(B_0) = \bigcap_{s \ge 0} \overline{\bigcup_{t \ge s} S(t) B_0} ,$$

where $B_0 = \left\{ (u, p, v, q) \in E_1 : \|(u, p, v, q)\|_{E_1}^2 = \|(u, v)\|_{V_{2m} \times V_{2m}}^2 + \|(p, q)\|_{V_m \times V_m}^2 \le c(R_0) + c(R_1) \right\}$, B_0 is the bounded absorbing set of E_1 and satisfies

(1) $S(t)A = A, t > 0$,

(2) $\lim_{t \to \infty} dist(S(t)B, A) = 0$, $\forall B \subset E_1$ is a bounded set,

where $dist(S(t)B, A) = \sup_{x \in B} \inf_{y \in A} \sup \inf \|S(t)x - y\|_{E_1}$.

Proof. Namely, we need to validate the conditions (1)-(3) of theorem 3.2.

Under the assumption of theorem 3.1, there exists the solution semigroup $S(t)$, here

$E_1 = V_{2m} \times V_m \times V_{2m} \times V_m, S(t): E_1 \to E_1.$

(1) According to theorem 2.1 and 2.2, we can acquire that $\forall B \subset E_1$ is a bounded set which includes in the ball $\left\{ \|(u, p, v, q)\|_{E_1} \le R \right\}$,

$$\|S(t)(u_0,p_0,v_0,q_0)\|_{E_1}^2=\|(u,v)\|_{V_{2m}\times V_{2m}}^2+\|(p,q)\|_{V_m\times V_m}^2$$
$$\le\|(u_0,v_0)\|_{V_{2m}\times V_{2m}}^2+\|(p_0,q_0)\|_{V_m\times V_m}^2+C\le R^2+C \quad \left(t\ge 0,(u_0,p_0,v_0,q_0)\in B\right),$$

it shows that $S(t)(t>0)$ is uniformly bounded in E_1.

(2) Furthemore, for any $(u_0,p_0,v_0,q_0)\in E_1$, since $t\ge\max\{t_0,t_1\}$, we have

$$\|S(t)(u_0,p_0,v_0,q_0)\|_{E_1}^2=\|(u,v)\|_{V_{2m}\times V_{2m}}^2+\|(p,q)\|_{V_m\times V_m}^2\le c(R_0)+c(R_1),$$

So we get B_0 is the bounded absorbing set.

(3) Due to $E_1\to E_0$ is the compact embedded, which means that the bounded set in E_1 is the compact set in E_0, so the semigroup operator $S(t)$ is completely continuous.

Hence, the semigroup operator $S(t)$ possesses a compact global attractor A.

The proof of theorem 3.3 is completed.

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