THE GLOBAL ATTRACTOR FOR THE KIRCHHOFF-TYPE EQUATIONS WITH STRONGLY DAMPED TERMS AND SOURCE TERMS

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ABSTRACT

The paper studies the initial boundary value problem for a class of the Kirchhoff-type equations with strongly damped terms and source terms in abounded domain. Under suitable assumptions, some priori estimates are acquired. In addition, the existence and uniqueness of the solutions are got by the Galerkin's method. Furthermore, the existence of the global attractor in $H_0^1(\Omega)\times H_0^m(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ is obtained.

Keywords: Kirchhoff-type equations; Galerkin's method; The global attractor; Existence; Uniqueness
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1. INTRODUCTION

In this paper, we consider the existence of the global attractor for the following Kirchhoff-type equations:

$$u_{tt}-M(\|\nabla u\|^2+\|\nabla^m v\|^2)\Delta u-\beta\Delta u_t+g_1(u,v)=f_1(x), \qquad (1.1)$$

$$v_{tt}+M(\|\nabla u\|^2+\|\nabla^m v\|^2)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \qquad (1.2)$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\Omega, \qquad (1.3)$$

$$v(x,0)=v_0(x), v_t(x,0)=v_1(x), \quad x\in\Omega, \qquad (1.4)$$

$$u(x,t)=v(x,t)=0, \quad x\in\partial\Omega, t>0, \qquad (1.5)$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\beta>0$ is a constant and $f_i(x)(i=1,2)$ is a given source term. Moreover, $M(\|\nabla u\|^2+\|\nabla^m v\|^2)$ is a scalar function. Then the assumptions on M and $g_i(u,v)$ will be specified later.

Let's review some results with respect to the equations of Kirchhoff-type. As is well known, Kirchhoff Eq. (1.6)was first studied by Kirchhoff(1883)[1]who investigated the following nonlinear vibration of an elastic string for $\delta=f=0$:

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}+f; \quad 0\le x\le L, t\ge 0 \qquad (1.6)$$

where $u=u(x,t)$ is the lateral deflection at the space coordinate x and the time t, ρ the mass density, h the cross-section area, p_0 the initial axial tension, L the length, E the Young's modulus, δ the resistance modulus, and f the external force.

This kind of wave models has been studied by many scholars under different types of assumptions. Particularly, well-posedness issues for Kirchhoff-type models like (1.1) were

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studied intensively in recent years. The single wave equation of Kirchhoff-type with a strongly damped term of the form:

$$u_{tt}+M(\left\|A^{\frac{1}{2}}u\right\|^{2})Au+\delta Au_{t}=f(u)\quad in\Omega\times[0,+\infty), \qquad (1.7)$$

$$u(x,0)=u_{0}(x),u_{t}(x,0)=u_{1}(x),\quad \text{and } u(x,t)\big|_{\partial\Omega}=0, \qquad (1.8)$$

where $A=-\Delta$,is studied by Kosuke Ono [2] who proved the global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff-type with a strong dissipation by using the 'Modified potential well' and 'Concavity' methods. Before that, Hosoya and Yamada [3] researched (1.7) with $M(r)\geq m_{0}>0$ (non-degenerate type) and with a linearly damped term u_{t} instead of Au_{t} ,and testified the global existence of a unique solution with the small data condition in $D(A)\times D(A^{\frac{1}{2}})$ (see [4]). Meanwhile, when $M(r)=r^{\gamma}$ (degenerate type) and $f(u)\equiv 0$, the global existence of the solution under the small initial data in $D(A)\times D(A^{\frac{1}{2}})$ has been obtained by Nishihara and Yamada [5]. Then, some authors also studied the decay estimates about the above of solutions. (see [6,7]).

In [8], Yunlong Gao, Yuting Sun studied the longtime behavior of solution to the initial boundary value problem for a class of strongly damped Higher-order Kirchhoff type equations:

$$u_{tt}+(-\Delta)^{m}u_{t}+\left\|D^{m}u\right\|^{2q}(-\Delta)^{m}u+g(u)=f(x),\quad (x,t)\in\Omega\times[0,+\infty), \qquad (1.9)$$

$$u\big|_{\partial\Omega}=0,\ u(x,0)=u_{0}(x),\ u_{t}(x,0)=u_{1}(x). \qquad (1.10)$$

In [9],Yunlong Gao studied the longtime behavior of solution to the initial boundary value problem for a class of strongly damped Higher-order Kirchhoff type equations:

$$u_{tt}+(-\Delta)^{m}u_{t}+(\alpha+\beta\left\|\nabla^{m}u\right\|^{2})^{q}(-\Delta)^{m}u+g(u)=f(x),\quad (x,t)\in\Omega\times[0,+\infty),$$

(1.11)

$$u(x,0)=u_{0}(x),u_{t}(x,0)=u_{1}(x),\quad x\in\Omega, \qquad (1.12)$$

$$u(x,t)=0,\frac{\partial^{i}u}{\partial v^{i}}=0,\quad i=1,2,\ldots,m-1,x\in\partial\Omega,t\in(0,+\infty). \qquad (1.13)$$

Under the suitable assumptions, Han and Wang [10] obtained the global existence and finite time

blow-up of the solutions to the system of the nonlinear viscoelastic wave equations:

$$u_{tt}-\Delta u+\int_{0}^{t}g_{1}(t-s)\Delta u(s)ds+\left|u_{t}\right|^{m-1}u_{t}=f_{1}(u,v),\qquad (x,t)\in\Omega\times(0,T), \qquad (1.14)$$

$$v_{tt}-\Delta v+\int_{0}^{t}g_{2}(t-s)\Delta v(s)ds+\left|v_{t}\right|^{r-1}v_{t}=f_{2}(u,v),\qquad (x,t)\in\Omega\times(0,T).$$

(1.15)

Then, Wu [11] improves the earlier blow-up results in the literature. He proved that under proper conditions on the nonlinearity of the damped and the source terms and certain initial data in the stable set and for a wider class of relaxation functions, the decay estimates of the energy function of exponential or polynomial depending on the exponents of the damping terms in both equations by Nakao's method. The equations as follows:

$$u_{tt}-M(\left\|\nabla u\right\|_{2}^{2}+\left\|\nabla v\right\|_{2}^{2})\Delta u+\int_{0}^{t}g_{1}(t-s)\Delta u(s)ds+\left|u_{t}\right|^{p-1}u_{t}=f_{1}(u,v), \qquad (1.16)$$

$$v_{tt}-M(\left\|\nabla u\right\|_{2}^{2}+\left\|\nabla v\right\|_{2}^{2})\Delta v+\int_{0}^{t}g_{2}(t-s)\Delta v(s)ds+\left|v_{t}\right|^{q-1}v_{t}=f_{2}(u,v). \qquad (1.17)$$

Motivated by previous results, it is interesting to study the global existence of solutions and the finite dimensional attractor of the problem (1.1)-(1.5).At first, we prove the global existence and uniqueness of the solution under suitable assumptions on the function M, $g_i(u,v)(i=1,2)$ and certain initial data. After that, we obtain the attractor of the equations.

The paper is arranged as follows: In Section2, we show that the preliminaries and some lemmas. In Section3, the existence, uniqueness of the solutions and the attractor are derived.

2.Preliminaries

In this paper, we introduce material needed in the proof our major results. The standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their inner product and norms are used in this section. Meanwhile we define $H_0^m(\Omega)=\left\{u\in H^m(\Omega):\frac{\partial^i u}{\partial v^i}=0, i=0,1,\cdots,m-1\right\}$ and introduce the abbreviations:

$E_0=H_0^1(\Omega)\times H_0^m(\Omega)\times L^2(\Omega)\times L^2(\Omega), E_1=(H^2(\Omega)\cap H_0^1(\Omega))\times(H^{2m}(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times H_0^m(\Omega),$

$\|\cdot\|=\|\cdot\|_{L^2(\Omega)}, \|\cdot\|_{L^q}=\|\cdot\|_{L^q(\Omega)}$ for any real number $q>1$.Next, we give some assumptions and notations needed in the proof of our results.

(H_1) Setting $g_1(u,v)=\frac{\partial G(u,v)}{\partial u}$, $g_2(u,v)=\frac{\partial G(u,v)}{\partial v}$ and $F(u,v)$ satisfying

$$F(u,v)\geq-\gamma(\|\nabla u\|^2+\|\nabla^m v\|^2)-C(\gamma), \qquad (2.1)$$

for $\forall\gamma>0$,there exist $C(\gamma)\geq 0$, and where $F(u,v)=\int_\Omega G(u,v)dx$.

(H_2) There exist constant $c_0>0$,such that

$$(g_1(u,v),u)+(g_2(u,v),v)\geq c_0F(u,v). \qquad (2.2)$$

(H_3) There exist constant $\beta>0$,such that

$$M(\|\nabla u\|^2+\|\nabla^m v\|^2)\geq\beta>0, \qquad (2.3)$$

and

$$M(\|\nabla u\|^2+\|\nabla^m v\|^2)(\|\nabla u\|^2+\|\nabla^m v\|^2)\geq\int_0^{(\|\nabla u\|^2+\|\nabla^m v\|^2)}M(s)ds. \qquad (2.4)$$

(H_4) There exist constant $c_1>0$,such that

$$|g_i(u,v)|\leq c_1(1+|u|^s+|v|^r), and\ g_i(u,v)\in C^2(\Omega), \ (i=1,2), \qquad (2.5)$$

where $2\leq r,s\leq\frac{2n}{n-2}(n>2)$ and $2\leq r,s\leq\infty(n\leq 2)$.

Lemma2.1.(Sobolev-Poincare inequality [12])Let s be a number with $2\leq s\leq\infty, n\leq 2m$ and $2\leq s\leq\frac{2m}{n-2m}, n>2m$.Then there is a constant K depending on Ω and s such that

$$\|u\|_s\leq K\left\|(-\Delta)^{\frac{m}{2}}u\right\|, \forall u\in H_0^m(\Omega). \qquad (2.6)$$

Lemma2.2. (Young's Inequality [13])For $\forall\varepsilon>0$ and $a,b\geq 0$,then

$$ab\leq\frac{\varepsilon^p}{p}a^p+\frac{1}{q\varepsilon^q}, \qquad (2.7)$$

where $\frac{1}{p}+\frac{1}{q}=1, p>1, q>1$.

Lemma2.3.(Gronwall's inequality [13])If $\forall t\in[t_0,+\infty), y(t)\ge 0$ and $\frac{dy}{dt}+gy\le h$,such that

$$y(t)\le y(t_0)e^{-g(t-t_0)}+\frac{h}{g}, t\ge t_0, \qquad (2.8)$$

where $g>0, h\ge 0$ are constants.

Lemma2.4. [14] Let ψ be an absolutely continuous positive function on R^+,which satisfies for some $\kappa>0$ the differential inequality

$$\frac{d}{dt}\psi(t)+2\kappa\psi(t)\le g(t)\psi(t)+h(t), \qquad t>0, \qquad (2.9)$$

where $h\in L^1_{loc}R^+$ and

$$\int_\tau^t g(y)dy\le\kappa(t-\tau)+m, \qquad for \quad t\ge\tau\ge 0, \qquad (2.10)$$

With some $m>0$.Then

$$\psi(t)\le e^m(\psi(s)e^{-\kappa(t-s)}+\int_s^t|h(y)|e^{-\kappa(t-y)}dy), \qquad \forall t\ge s\ge 0. \qquad (2.11)$$

Lemma2.5. Assume $(H_1)\sim(H_3)$ hold, and $(u_0,v_0,p_0,q_0)\in E_0$, $f_i(x)\in L^2(\Omega)$ **.**Since the solution (u,v,p,q) of the problem(1.1)-(1.5)satisfies $(u,v,p,q)\in E_0$,and

$$\|\nabla u\|^2+\|\nabla^m v\|^2+\|p\|^2+\|q\|^2\le\frac{y_1(0)\cdot e^{-c_1\varepsilon_1 t}}{\min\left\{1,\frac{3\beta}{4}\right\}}+\frac{C_2}{c_1\varepsilon_1\min\left\{1,\frac{3\beta}{4}\right\}}, \qquad (2.12)$$

where $p=u_t+\varepsilon_1 u, q=v_t+\varepsilon_1 v, p_o=u_1+\varepsilon_1 u_0, q_0=v_1+\varepsilon_1 v_0$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, $c_1\varepsilon_1=\min\left\{-\varepsilon_1^{\ 2}-(\frac{B\lambda_1+2}{2})\varepsilon_1+\beta\lambda_1, -\varepsilon_1^{\ 2}-(\frac{B\lambda_1^{\ m}+2}{2})\varepsilon_1+\beta\lambda_1^{\ m}, \frac{\varepsilon_1}{4}, \frac{c_0}{2}\right\}$,

$$0<\varepsilon_1<\min\{\frac{\beta\lambda_1}{2}, \frac{\beta\lambda_1^m}{2},$$
$$\frac{-(\beta\lambda_1+2)+\sqrt{(\beta\lambda_1+2)^2+16\beta\lambda_1}}{4}, \frac{-(\beta\lambda_1^m+2)+\sqrt{(\beta\lambda_1^m+2)^2+16\beta\lambda_1^m}}{4}\},$$

and $c_2=c_2(\|f_1(x)\|^2, \|f_2(x)\|^2, C(\gamma_1))$,

$y_1(0)=\|p_0\|^2+\|q_0\|^2+\int_0^{\|\nabla u_0\|^2+\|\nabla^m v_0\|^2}M(s)ds+2F(u_0,v_0)+2C(\gamma_1)$, $\gamma_1=\frac{\beta}{8}$ So, there exists R_0 and $t_0=t_0(\Omega)>0$,such that

$$\|(u,v,p,q)\|_{E_0}^2=\|\nabla u\|^2+\|\nabla^m v\|^2+\|p\|^2+\|q\|^2\le R_0^2, (t>t_0). \qquad (2.13)$$

Proof. Multiplying Eq.(1.1)by $p=u_t+\varepsilon_1 u$,then

$$(u_{tt}-M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ \Delta u-\beta\Delta u_t+g_1(u,v),p)=(f_1(x),p). \qquad (2.14)$$

By using Holder inequality, Poincare inequality, Young's inequality, we obtain

$$\begin{aligned}(u_{tt},p)&=\frac{1}{2}\frac{d}{dt}\|p\|^2-\varepsilon_1\|p\|^2+(\varepsilon_1^{\ 2}u,p)\\&\ge\frac{1}{2}\frac{d}{dt}\|p\|^2-\frac{(2\varepsilon_1+\varepsilon_1^{\ 2})}{2}\|p\|^2-\frac{\varepsilon_1^{\ 2}}{2\lambda_1}\|\nabla u\|^2\end{aligned}. \qquad (2.15)$$

$$\begin{aligned}&(-M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ \Delta u, p)\\&=M(\|\nabla u\|^2+\|\nabla^m v\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\varepsilon_1 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\nabla u\|^2\end{aligned}. \tag{2.16}$$

$$\begin{aligned}(-\beta\Delta u_t, p)=(-\beta\Delta(p-\varepsilon_1 u), p)&=\beta\|\nabla p\|^2-\beta\varepsilon_1(\nabla u,\nabla p)\\&\geq\frac{\beta(2-\varepsilon_1)}{2}\|\nabla p\|^2-\frac{\beta\varepsilon_1}{2}\|\nabla u\|^2\\&\geq\frac{\lambda_1\beta(2-\varepsilon_1)}{2}\|p\|^2-\frac{\beta\varepsilon_1}{2}\|\nabla u\|^2\end{aligned}. \tag{2.17}$$

$$\begin{aligned}(g_1(u,v), p)&=(g_1(u,v),u_t)+(g_1(u,v),\varepsilon_1 u)\\&=\int_\Omega g_1(u,v)u_t dx+\varepsilon_1(g_1(u,v),u)\end{aligned}. \tag{2.18}$$

$$(f_1(x), p)\leq\frac{\varepsilon_1^{\ 2}}{2}\|p\|^2+\frac{1}{2\varepsilon_1^{\ 2}}\|f_1(x)\|^2. \tag{2.19}$$

Plug (2.15) into (2.19), then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|p\|^2+[-\varepsilon_1^{\ 2}-(\frac{B\lambda_1+2}{2})\varepsilon_1+\beta\lambda_1]\|p\|^2-(\frac{\varepsilon_1^{\ 2}}{2\lambda_1}+\frac{\beta\varepsilon_1}{2})\|\nabla u\|^2\\&+M(\|\nabla u\|^2+\|\nabla^m v\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\varepsilon_1 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\nabla u\|^2\\&+\int_\Omega g_1(u,v)u_t dx+\varepsilon_1(g_1(u,v),u)\leq\frac{\|f_1(x)\|^2}{2\varepsilon_1^{\ 2}}\end{aligned}. \tag{2.20}$$

Multiplying Eq. (1.2) by $q=v_t+\varepsilon_1 v$, then

$$(v_{tt}+M(\|\nabla u\|^2+\|\nabla^m v\|^2)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v),q)=(f_2(x),q). \tag{2.21}$$

By using Holder inequality, Poincare inequality, Young's inequality, we obtain

$$\begin{aligned}(v_{tt}, p)&=\frac{1}{2}\frac{d}{dt}\|q\|^2-\varepsilon_1\|q\|^2+(\varepsilon_1^{\ 2}v,q)\\&\geq\frac{1}{2}\frac{d}{dt}\|q\|^2-\frac{(2\varepsilon_1+\varepsilon_1^{\ 2})}{2}\|q\|^2-\frac{\varepsilon_1^{\ 2}}{2\lambda_1^{\ m}}\|\nabla^m v\|^2\end{aligned}. \tag{2.22}$$

$$\begin{aligned}&(M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ (-\Delta)^m v,q)\\&=M(\|\nabla u\|^2+\|\nabla^m v\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla^m v\|^2+\varepsilon_1 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\nabla^m v\|^2\end{aligned}. \tag{2.23}$$

$$(\beta(-\Delta)^m v_t, q) = (\beta(-\Delta)^m (q-\varepsilon_1 v), q) = \beta\left\|\nabla^m q\right\|^2 - \beta\varepsilon_1(\nabla^m v, \nabla^m q)$$
$$\geq \frac{\beta(2-\varepsilon_1)}{2}\left\|\nabla^m q\right\|^2 - \frac{\beta\varepsilon_1}{2}\left\|\nabla^m v\right\|^2 .$$
$$\geq \frac{\lambda_1^m\beta(2-\varepsilon_1)}{2}\left\|q\right\|^2 - \frac{\beta\varepsilon_1}{2}\left\|\nabla^m v\right\|^2$$

(2.24)

$$(g_2(u,v), q) = (g_2(u,v), v_t) + (g_2(u,v), \varepsilon_1 v)$$
$$= \int_\Omega g_2(u,v)v_t dx + \varepsilon_1(g_2(u,v), v) \qquad (2.25)$$

$$(f_2(x), q) \leq \frac{\varepsilon_1^2}{2}\left\|q\right\|^2 + \frac{1}{2\varepsilon_1^2}\left\|f_2(x)\right\|^2 . \qquad (2.26)$$

Plug (2.22) into (2.26), then

$$\frac{1}{2}\frac{d}{dt}\left\|q\right\|^2 + [-\varepsilon_1^2 - (\frac{B\lambda_1^m + 2}{2})\varepsilon_1 + \beta\lambda_1^m]\left\|q\right\|^2 - (\frac{\varepsilon_1^2}{2\lambda_1^m} + \frac{\beta\varepsilon_1}{2})\left\|\nabla^m v\right\|^2$$
$$+M(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2)\frac{1}{2}\frac{d}{dt}\left\|\nabla^m v\right\|^2 + \varepsilon_1 M(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2)\left\|\nabla^m v\right\|^2 \quad . \qquad (2.27)$$
$$+\int_\Omega g_2(u,v)v_t dx + \varepsilon_1(g_2(u,v), v) \leq \frac{\left\|f_2(x)\right\|^2}{2\varepsilon_1^2}$$

Adding (2.20) to (2.27), by $(H_1), (H_2)$, we have

$$\frac{1}{2}\frac{d}{dt}(\left\|p\right\|^2 + \left\|q\right\|^2 + \int_0^{\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2} M(s)ds + 2F(u,v)) + [-\varepsilon_1^2 - (\frac{B\lambda_1 + 2}{2})\varepsilon_1 + \beta\lambda_1]\left\|p\right\|^2 +$$
$$[-\varepsilon_1^2 - (\frac{B\lambda_1^m + 2}{2})\varepsilon_1 + \beta\lambda_1^m]\left\|q\right\|^2 - (\frac{\varepsilon_1^2}{2\lambda_1} + \frac{\beta\varepsilon_1}{2})\left\|\nabla u\right\|^2 - (\frac{\varepsilon_1^2}{2\lambda_1^m} + \frac{\beta\varepsilon_1}{2})\left\|\nabla^m v\right\|^2 + \quad . \qquad (2.28)$$
$$\varepsilon_1 M(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2)(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2) + c_0 F(u,v) \leq \frac{\left\|f_1(x)\right\|^2 + \left\|f_2(x)\right\|^2}{2\varepsilon_1^2}$$

By (2.1), (H_3), we get

$$\frac{1}{4}\int_0^{\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2} M(s)ds + 2F(u,v) + 2C(\gamma_1)$$
$$\geq 2(\frac{\beta}{8}(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2) + F(u,v) + C(\gamma_1) \geq 0 \quad , \qquad (2.29)$$

where $\gamma_1 = \frac{\beta}{8}$.

By (H_3), we have

$$\frac{3}{4}\varepsilon_1 M(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2)(\left\|\nabla u\right\|^2 + \left\|\nabla^m v\right\|^2) - (\frac{\varepsilon_1^2}{2\lambda_1} + \frac{\beta\varepsilon_1}{2})\left\|\nabla u\right\|^2 - (\frac{\varepsilon_1^2}{2\lambda_1^m} + \frac{\beta\varepsilon_1}{2})\left\|\nabla^m v\right\|^2$$
$$\geq (\frac{3\beta\varepsilon_1}{4} - \frac{\varepsilon_1^2}{2\lambda_1} - \frac{\beta\varepsilon_1}{2})\left\|\nabla u\right\|^2 + (\frac{3\beta\varepsilon_1}{4} - \frac{\varepsilon_1^2}{2\lambda_1^m} - \frac{\beta\varepsilon_1}{2})\left\|\nabla^m v\right\|^2 \geq 0 \quad . \qquad (2.30)$$

Therefore, from (2.28), (2.30), there exist a constant $c_1 > 0$,satisfies

$$c_1\varepsilon_1 = \min\left\{-\varepsilon_1{}^2 - (\frac{B\lambda_1 + 2}{2})\varepsilon_1 + \beta\lambda_1, -\varepsilon_1{}^2 - (\frac{B\lambda_1{}^m + 2}{2})\varepsilon_1 + \beta\lambda_1{}^m, \frac{\varepsilon_1}{4}, \frac{c_0}{2}\right\}, \qquad (2.31)$$

such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|p\|^2 + \|q\|^2 + \int_0^{\|\nabla u\|^2+\|\nabla^m v\|^2} M(s)ds + 2F(u,v) + 2C(\gamma_1)) + \\ &c_1\varepsilon_1[\|p\|^2 + \|q\|^2 + \int_0^{\|\nabla u\|^2+\|\nabla^m v\|^2} M(s)ds + 2F(u,v) + 2C(\gamma_1)] \le c_2\end{aligned}, \qquad (2.32)$$

where $c_2 = c_2(\|f_1(x)\|^2, \|f_2(x)\|^2, C(\gamma_1))$.

Then, we set $y_1(t) = \|p\|^2 + \|q\|^2 + \int_0^{\|\nabla u\|^2+\|\nabla^m v\|^2} M(s)ds + 2F(u,v) + 2C(\gamma_1)$,(2.32) is simplified as

$$\frac{d}{dt}y_1(t) + c_1\varepsilon_1 y_1(t) \le c_2. \qquad (2.33)$$

With (2.29),we can get $y_1(t) \ge 0$.Hence, by using Gronwall's inequality, we have

$$\|\nabla u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2 \le y_1(t) \le \frac{y_1(0)\cdot e^{-c_1\varepsilon_1 t}}{\min\left\{1, \frac{3\beta}{4}\right\}} + \frac{C_2}{c_1\varepsilon_1 \min\left\{1, \frac{3\beta}{4}\right\}}. \qquad (2.34)$$

Next,

$$\varlimsup_{t\to\infty}\|(u,v,p,q)\|_{E_0}^2 = \|\nabla u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2 \le \frac{C_2}{c_1\varepsilon_1 \min\left\{1, \frac{3\beta}{4}\right\}}. \qquad (2.35)$$

So, there exist R_0 and $t_0 = t_0(\Omega) > 0$,such that

$$\|(u,v,p,q)\|_{E_0}^2 = \|\nabla u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2 \le R_0^2, (t > t_0). \qquad (2.36)$$

Lemma2.6.

$$(H_5): 2((f_1,u) + (f_2,v)) \le 2c_0, \|u_t\|^2 + \|v_t\|^2 + \int_0^{\|\nabla u\|^2+\|\nabla^m v\|^2} M(\|\nabla u\|^2 + \|\nabla^m v\|^2)dt + 2F(u,v) - 2c_0 > 0$$

Under the hypothesis of Lemma2.5., $(H_1)-(H_4)$ hold and $(u_0,v_0,p_0,q_0) \in E_1$, $f_1(x) \in H_0^1(\Omega)$, $f_2(x) \in H_0^m(\Omega)$.Then the solution (u,v,p,q) of the problem(1.1)-(1.5)satisfies $(u,v,p,q) \in E_1$,and

$$\|\Delta u\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le \frac{e^m y_2(0)\cdot e^{-c_8\varepsilon_2 t}}{\min\{1,\beta\}} + \frac{e^m c_9}{c_8\varepsilon_2 \min\{1,\beta\}}, \qquad (2.37)$$

where $p = u_t + \varepsilon_2 u, q = v_t + \varepsilon_2 v, p_o = u_1 + \varepsilon_2 u_0, q_0 = v_1 + \varepsilon_2 v_0$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$,and, $c_9 = c_9(c_6, \|\nabla f_1(x)\|^2, \|\nabla f_2(x)\|^2)$,

$$0 < \varepsilon_1 < \min\{\beta, \frac{\beta\lambda_1}{2}, \frac{-(\beta\lambda_1 + 2) + \sqrt{(\beta\lambda_1 + 2)^2 + 8\beta\lambda_1}}{4},$$

$$\frac{\beta\lambda_1^m}{2}, \frac{-(\beta\lambda_1^m + 2) + \sqrt{(\beta\lambda_1^m + 2)^2 + 8\beta\lambda_1^m}}{4}\},\ c_8\varepsilon_2 = \min\{-\varepsilon_2{}^2 - (\frac{B\lambda_1 + 2}{2})\varepsilon_2 + \frac{\beta\lambda_1}{2}, \frac{\varepsilon_2}{4},$$

$$-\varepsilon_2{}^2 - (\frac{B\lambda_1^m + 2}{2})\varepsilon_2 + \frac{\beta\lambda_1^m}{2}\}, y_2(0) = \|\nabla p_0\|^2 + \|\nabla^m q_0\|^2 + M(\|\nabla u\|^2 + \|\nabla^m v\|^2)(\|\Delta u\|^2 + \|\Delta^m v\|^2),$$

So, there exists R_1 and $t_1 = t_1(\Omega) > 0$,such that

$$\|(u,v,p,q)\|_{E_1}^2 = \|\Delta \mathrm{u}\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le R_1^2, (t > t_1)\,. \qquad (2.38)$$

Proof. Taking inner product by $(-\Delta)p = (-\Delta)u_t + (-\Delta)\varepsilon_2 u$,in(1.1),we obtain

$$(u_{tt} - M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\ \Delta u - \beta\Delta u_t + g_1(u,v), (-\Delta)p) = (f_1(x), (-\Delta)p)\,. \qquad (2.39)$$

Following the computation in (2.39) one be one and using Poincare's inequality, Young's inequality, we have

$$\begin{aligned}(u_{tt}, (-\Delta)p) &= \frac{1}{2}\frac{d}{dt}\|\nabla p\|^2 - \varepsilon_2\|\nabla p\|^2 + (\varepsilon_2{}^2\nabla u, \nabla p) \\ &\ge \frac{1}{2}\frac{d}{dt}\|\nabla p\|^2 - \frac{(2\varepsilon_2 + \varepsilon_2{}^2)}{2}\|\nabla p\|^2 - \frac{\varepsilon_2{}^2}{2\lambda_1}\|\Delta \mathrm{u}\|^2\end{aligned}. \qquad (2.40)$$

$$\begin{aligned}&(-M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\ \Delta u, (-\Delta)p) \\ &= (-M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\ \Delta u, (-\Delta)u_t) + (-M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\ \Delta u, (-\Delta)\varepsilon_2 u) \\ &= \frac{1}{2}\frac{d}{dt}[M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\|\Delta u\|^2] + \varepsilon_2 M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\|\Delta u\|^2 - \frac{\|\Delta u\|^2}{2}\frac{d}{dt}M(\|\nabla \mathrm{u}\|^2 + \|\nabla^m v\|^2)\end{aligned}.$$

(2.41)

$$\begin{aligned}(-\beta\Delta u_t, (-\Delta)p) &= \beta\|\Delta p\|^2 - \beta\varepsilon_2(\Delta u, \Delta p) \\ &\ge \frac{\beta(2-\varepsilon_2)}{2}\|\Delta p\|^2 - \frac{\beta\varepsilon_2}{2}\|\Delta \mathrm{u}\|^2\end{aligned}. \qquad (2.42)$$

By using (H_4) ,we can easily find a constant $c_3 > 0$,such that

$$\begin{aligned}\|g_i(u,v)\|^2 = \int_\Omega |g_i(u,v)|^2 dx &\le \int_\Omega c_1(1+|u|^s + |v|^r)^2 dx \\ &\le c_3(1+\|u\|_{L^{2s}}^{2s} + \|v\|_{L^{2r}}^{2r})\end{aligned}. \qquad (2.43)$$

Next, we need the further estimate of (2.43).By the interpolation inequality and Lemma2.5.,we have

$$\|u\|_{L^{2s}}^{2s} \le c_4\|\nabla \mathrm{u}\|^{n(s-1)}\|\mathrm{u}\|^{2s-n(s-1)} \qquad for\begin{cases}1\le s\le \dfrac{n}{n-2} & (n>2)\\ 1\le s\le\infty & (n\le 2)\end{cases}, \qquad (2.44)$$

$$\|v\|_{L^{2r}}^{2r} \le c_5\|\nabla^m v\|^{\frac{n(r-1)}{m}}\|v\|^{\frac{2r-n(r-1)}{m}} \qquad for\begin{cases}1\le r\le \dfrac{n}{n-2m} & (n>2m)\\ 1\le r\le\infty & (n\le 2m)\end{cases}. \qquad (2.45)$$

So, there exist $c_6 > 0$,such that

$$\|g_i(u,v)\|^2 \le 2\beta c_6 \qquad (2.46)$$

Furthermore, we have

$$(g_1(u,v), (-\Delta)p) \ge -\frac{1}{2\beta}\|g_1(u,v)\|^2 - \frac{\beta}{2}\|\Delta p\|^2\,. \qquad (2.47)$$

$$(f_1(x), (-\Delta)p) \le \frac{\varepsilon_2{}^2}{2}\|\nabla p\|^2 + \frac{\|\nabla f_1(x)\|^2}{2\varepsilon_2{}^2}\,. \qquad (2.48)$$

Substituting (2.40)-(2.42),(2,47)-(2.48),into(2.39), then

$$\frac{1}{2}\frac{d}{dt}[\|\nabla p\|^2+M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta u\|^2]+[-\varepsilon_2{}^2-(\frac{B\lambda_1+2}{2})\varepsilon_2+\frac{\beta\lambda_1}{2}]\|\nabla p\|^2$$
$$-(\frac{\varepsilon_2{}^2}{2\lambda_1}+\frac{\beta\varepsilon_2}{2})\|\Delta u\|^2+\varepsilon_2 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta u\|^2 .$$
$$\le\frac{\|\nabla f_1(x)\|^2}{2\varepsilon_2{}^2}+c_6+\frac{\|\Delta u\|^2}{2}\frac{d}{dt}M(\|\nabla u\|^2+\|\nabla^m v\|^2)$$

(2.49)

Similar to the above calculation, we use $(-\Delta)^m q=(-\Delta)^m v_t+(-\Delta)^m\varepsilon_2 v$ multiply with both sides of Eq.(1.2),and we get

$$(v_{tt}+M(\|\nabla u\|^2+\|\nabla^m v\|^2)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v),(-\Delta)^m q)=(f_2(x),(-\Delta)^m q).$$

(2.50)

Then,

$$(v_{tt},(-\Delta)^m q)=\frac{1}{2}\frac{d}{dt}\|\nabla^m q\|^2-\varepsilon_2\|\nabla^m q\|^2+(\varepsilon_2{}^2\nabla^m v,\nabla^m q)$$
$$\ge\frac{1}{2}\frac{d}{dt}\|\nabla^m q\|^2-\frac{(2\varepsilon_2+\varepsilon_2{}^2)}{2}\|\nabla^m q\|^2-\frac{\varepsilon_2{}^2}{2\lambda_1{}^m}\|\nabla^m v\|^2 . \quad (2.51)$$

$$(M(\|\nabla u\|^2+\|\nabla^m v\|^2)(-\Delta)^m v,(-\Delta)^m q)$$
$$=\frac{1}{2}\frac{d}{dt}[M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta^m v\|^2]+\varepsilon_2 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta^m v\|^2 . \quad (2.52)$$
$$-\frac{\|\Delta^m v\|^2}{2}\frac{d}{dt}M(\|\nabla u\|^2+\|\nabla^m v\|^2)$$

$$(-\beta(-\Delta)^m v_t,(-\Delta)^m q)=\beta\|\Delta^m q\|^2-\beta\varepsilon_2(\Delta^m v,\Delta^m q)$$
$$\ge\frac{\beta(2-\varepsilon_2)}{2}\|\Delta^m q\|^2-\frac{\beta\varepsilon_2}{2}\|\Delta^m v\|^2 . \quad (2.53)$$

$$(g_2(u,v),(-\Delta)^m q)\ge-\frac{1}{2\beta}\|g_2(u,v)\|^2-\frac{\beta}{2}\|\Delta^m q\|^2 . \quad (2.54)$$

$$(f_2(x),(-\Delta)^m q))\le\frac{\varepsilon_2{}^2}{2}\|\nabla^m q\|^2+\frac{\|\nabla^m f_2(x)\|^2}{2\varepsilon_2{}^2} . \quad (2.55)$$

From the above, we obtain

$$\frac{1}{2}\frac{d}{dt}[\|\nabla^m q\|^2+M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta^m v\|^2]+[-\varepsilon_2{}^2-(\frac{B\lambda_1{}^m+2}{2})\varepsilon_2+\frac{\beta\lambda_1{}^m}{2}]\|\nabla^m q\|^2$$
$$-(\frac{\varepsilon_2{}^2}{2\lambda_1{}^m}+\frac{\beta\varepsilon_2}{2})\|\Delta^m v\|^2+\varepsilon_2 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\|\Delta^m v\|^2 . \quad (2.56)$$
$$\le\frac{\|\nabla f_2(x)\|^2}{2\varepsilon_2{}^2}+c_6+\frac{\|\Delta^m v\|^2}{2}\frac{d}{dt}M(\|\nabla u\|^2+\|\nabla^m v\|^2)$$

Plug (2.49) into (2.56), we get

$$\frac{1}{2}\frac{d}{dt}[\|\nabla p\|^2+\|\nabla^m q\|^2+M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ (\|\Delta u\|^2+\|\Delta^m v\|^2)]+[-\varepsilon_2{}^2-(\frac{B\lambda_1+2}{2})\varepsilon_2+\frac{\beta\lambda_1}{2}]\|\nabla p\|^2$$
$$+[-\varepsilon_2{}^2-(\frac{B\lambda_1{}^m+2}{2})\varepsilon_2+\frac{\beta\lambda_1{}^m}{2}]\|\nabla^m q\|^2-(\frac{\varepsilon_2{}^2}{2\lambda_1}+\frac{\beta\varepsilon_2}{2})\|\Delta u\|^2-(\frac{\varepsilon_2{}^2}{2\lambda_1{}^m}+\frac{\beta\varepsilon_2}{2})\|\Delta^m v\|^2$$
$$+\varepsilon_2 M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ (\|\Delta u\|^2+\|\Delta^m v\|^2)$$
$$\leq\frac{(\|\Delta u\|^2+\|\Delta^m v\|^2)}{2}\frac{d}{dt}M(\|\nabla u\|^2+\|\nabla^m v\|^2)+\frac{\|\nabla f_1(x)\|^2+\|\nabla^m f_2(x)\|^2}{2\varepsilon_2{}^2}+2c_6 .$$

(2.57)

Similar to Lemma2.5., we have

$$\frac{3}{4}\varepsilon_2 M(\|\nabla u\|^2+\|\nabla^m v\|^2)(\|\Delta u\|^2+\|\Delta^m v\|^2)-(\frac{\varepsilon_2{}^2}{2\lambda_1}+\frac{\beta\varepsilon_2}{2})\|\Delta u\|^2-(\frac{\varepsilon_2{}^2}{2\lambda_1{}^m}+\frac{\beta\varepsilon_2}{2})\|\Delta^m v\|^2$$
$$\geq(\frac{3\beta\varepsilon_2}{4}-\frac{\varepsilon_1{}^2}{2\lambda_1}-\frac{\beta\varepsilon_2}{2})\|\Delta u\|^2+(\frac{3\beta\varepsilon_2}{4}-\frac{\varepsilon_2{}^2}{2\lambda_1{}^m}-\frac{\beta\varepsilon_2}{2})\|\Delta^m v\|^2\geq 0 .$$

(2.58)

Next, by (H_3),as following

$$\frac{\varepsilon_2}{4}M(\|\nabla u\|^2+\|\nabla^m v\|^2)(\|\Delta u\|^2+\|\Delta^m v\|^2)\geq\frac{\beta\varepsilon_2}{4}(\|\Delta u\|^2+\|\Delta^m v\|^2) \qquad (2.59)$$

Setting $y_2(t)=\|\nabla p\|^2+\|\nabla^m q\|^2+M(\|\nabla u\|^2+\|\nabla^m v\|^2)(\|\Delta u\|^2+\|\Delta^m v\|^2)$, which imply that

$$\|\Delta u\|^2+\|\Delta^m v\|^2\leq\frac{M(\|\nabla u\|^2+\|\nabla^m v\|^2)\ (\|\Delta u\|^2+\|\Delta^m v\|^2)}{\beta}\leq\frac{y_2(t)}{\beta} . \qquad (2.60)$$

By Lemma2.5.,

$$\frac{d}{dt}M(\|\nabla u\|^2+\|\nabla^m v\|^2)\leq c_7(\|\nabla u_t\|+\|\nabla^m v_t\|) . \qquad (2.61)$$

Substituting (2.58),(2.60)-(2.61) into (2.57), then

$$\frac{d}{dt}y_2(t)+2c_8\varepsilon_2 y_2(t)\leq\frac{c_7}{2\beta}(\|\nabla u_t\|+\|\nabla^m v_t\|)y_2(t)+c_9 , \qquad (2.62)$$

where $,c_8\varepsilon_2=\min\left\{-\varepsilon_2{}^2-(\frac{B\lambda_1+2}{2})\varepsilon_2+\frac{\beta\lambda_1}{2},-\varepsilon_2{}^2-(\frac{B\lambda_1{}^m+2}{2})\varepsilon_2+\frac{\beta\lambda_1{}^m}{2},\frac{\varepsilon_2}{4}\right\}$,

$c_9=c_9(c_6,\|\nabla f_1(x)\|^2,$
$\|\nabla f_2(x)\|^2)$.

Moreover, taking the inner product of (1.1),(1.2)with u_t, v_t ,we obtain

$$\int_t^{+\infty}(\|\nabla u_t\|+\|\nabla^m v_t\|)ds\leq c_{10} , \qquad (2.63)$$

where $c_{10}>0$ is a constant.

Therefore, with some $m>0$,we have

$$\int_0^t\frac{c_7}{2\beta}(\|\nabla u_t\|+\|\nabla^m v_t\|)ds\leq m+c_8\varepsilon_2 t \quad for \quad \forall t\geq 0 . \qquad (2.64)$$

By Lemma2.5., then

$$\begin{aligned} y_2(t) &\le e^m y_2(0)\cdot e^{-c_8\varepsilon_2 t} + e^m\int_0^t c_9 e^{-c_8\varepsilon_2 t}ds \qquad for \quad \forall t\ge 0 \\ &\le e^m y_2(0)\cdot e^{-c_8\varepsilon_2 t} + \frac{e^m c_9}{c_8\varepsilon_2} \end{aligned} . \qquad (2.65)$$

So,

$$\|\Delta u\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le \frac{e^m y_2(0)\cdot e^{-c_8\varepsilon_2 t}}{\min\{1,\beta\}} + \frac{e^m c_9}{c_8\varepsilon_2 \min\{1,\beta\}} .$$

(2.66)

Then,

$$\overline{\lim_{t\to\infty}}\|(u,v,p,q)\|_{E_1}^2 = \|\Delta u\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le \frac{e^m c_9}{c_8\varepsilon_2 \min\{1,\beta\}} .$$

(2.67)

Therefore, there exists R_1 and $t_1 = t_1(\Omega) > 0$,such that

$$\|(u,v,p,q)\|_{E_1}^2 = \|\Delta u\|^2 + \|\Delta^m v\|^2 + \|\nabla p\|^2 + \|\nabla^m q\|^2 \le R_1^2, (t > t_1) .$$

(2.68)

3.Global Attractor

3.1. The Existence and Uniqueness of Solution

Theorem3.1. Assume $(H_1)-(H_4)$ hold, $(H_6): 0 < k_0 \le \frac{d}{dt} M(\|\nabla u\|^2 + \|\nabla^m v\|^2) \le k_1$, and $(u_0, v_0, p_0, q_0) \in E_1$, $f_1(x) \in H_0^1(\Omega)$, $f_2(x) \in H_0^m(\Omega)$. $p = u_t + \varepsilon u$, $q = v_t + \varepsilon v$. Hence, Eq. (1.1)-(1.5)
exists a unique smooth solution

$$(u,v,p,q) \in L^\infty((0,\infty); E_1) . \qquad (3.1)$$

Proof. Using the Galerkin method, Lemma2.5. and Lemma2.6., we can get the existence of solutions. Now, let's prove the uniqueness of solutions.

Suppose $w_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, w_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ are two solutions of the problems (1.1)-(1.5).Let $w = \begin{pmatrix} u \\ v \end{pmatrix} = w_1 - w_2$. Then, $w(x,0) = w_0(x) = 0, w_t(x,0) = w_1(x) = 0$,by two equations subtract,we have

$$u_{tt} + M(\|\nabla u_2\|^2 + \|\nabla^m v_2\|^2)\Delta u_2 - M(\|\nabla u_1\|^2 + \|\nabla^m v_1\|^2)\Delta u_1 - \beta\Delta u_t + g_1(u_1,v_1) - g_1(u_2,v_2) = 0 ,$$

(3.2)

$$v_{tt} + M(\|\nabla u_1\|^2 + \|\nabla^m v_1\|^2)(-\Delta)^m v_1 - M(\|\nabla u_2\|^2 + \|\nabla^m v_2\|^2)(-\Delta)^m v_2 + \beta(-\Delta)^m v_t + g_2(u_1,v_1) - g_2(u_2,v_2) = 0$$

.(3.3)

By using u_t to inner product of Eq. (3.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + (M(\|\nabla u_2\|^2 + \|\nabla^m v_2\|^2)\Delta u_2 - M(\|\nabla u_1\|^2 + \|\nabla^m v_1\|^2)\Delta u_1, u_t) \\ &+\beta\|\nabla u_t\|^2 + (g_1(u_1,v_1) - g_1(u_2,v_2), u_t) = 0 \end{aligned} .$$

(3.4)

In the next section, we process items in (3.4), respectively.

By (H_6), we have

$$M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2=\frac{1}{2}\frac{d}{dt}[M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla u\|^2]-\frac{\|\nabla u\|^2}{2}\frac{d}{dt}M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)$$
$$\geq\frac{1}{2}\frac{d}{dt}[M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla u\|^2]-k_1\frac{\|\nabla u\|^2}{2}$$

. (3.5)

By Lagrange's mean value theorem, Lemma2.5. and Lemma2.6., we have

$$\left|(M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\Delta u_1-M(\|\nabla u_1\|^2+\|\nabla^m v_1\|^2)\Delta u_1,u_t)\right|\leq c_{12}(\|\nabla u\|+\|\nabla^m v\|)\|u_t\|$$
$$\leq c_{13}(\|\nabla u\|^2+\|\nabla^m v\|^2)+\frac{\|u_t\|^2}{2}.$$

(3.6)

Hence,

$$(M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\Delta u_2-M(\|\nabla u_1\|^2+\|\nabla^m v_1\|^2)\Delta u_1,u_t)$$
$$=M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+(M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\Delta u_1-M(\|\nabla u_1\|^2+\|\nabla^m v_1\|^2)\Delta u_1,u_t).$$
$$\geq\frac{1}{2}\frac{d}{dt}[M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla u\|^2]-\frac{k_1\|\nabla u\|^2}{2}-c_{13}(\|\nabla u\|^2+\|\nabla^m v\|^2)-\frac{\|u_t\|^2}{2}$$

(3.7)

By (H_4), and young's inequality,we get

$$|(g_1(u_1,v_1)-g_1(u_2,v_2),u_t)|=(g_1(u_1,v_1)-g_1(u_2,v_1)+g_1(u_2,v_1)-g_1(u_2,v_2),u_t)$$
$$=(g_{1u}(\xi_1,v_1)(u_1-u_2),u_t)+(g_{1v}(u_2,\xi_2)(v_1-v_2),u_t)$$
$$\leq c_{14}\|u\|\cdot\|u_t\|+c_{15}\|v\|\cdot\|u_t\|.$$
$$\leq\frac{c_{14}\|\nabla u\|^2}{2\lambda_1}+\frac{c_{15}\|\nabla^m v\|^2}{2\lambda_1^m}+\|u_t\|^2$$

(3.8)

Then,

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla u\|^2]$$
$$\leq(\frac{3}{2}-\lambda_1\beta)\|u_t\|^2+(c_{13}+\frac{k_1}{2}+\frac{c_{14}}{2\lambda_1})\|\nabla u\|^2+(c_{13}+\frac{c_{15}}{2\lambda_1^m})\|\nabla^m v\|^2. \quad (3.9)$$

Analogous to the above, we use v_t with both sides of Eq.(3.3)and obtain

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2+(M(\|\nabla u_1\|^2+\|\nabla^m v_1\|^2)(-\Delta)^m v_1-M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(-\Delta)^m v_2,v_t)$$
$$+\beta\|\nabla^m v_t\|^2+(g_2(u_1,v_1)-g_2(u_2,v_2),v_t)=0.$$

(3.10)

Next, we process items in (3.10), respectively

$$(M(\|\nabla u_1\|^2+\|\nabla^m v_1\|^2)(-\Delta)^m v_1-M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(-\Delta)^m v_2,v_t)$$
$$\geq\frac{1}{2}\frac{d}{dt}[M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla^m v\|^2]-\frac{k_1\|\nabla^m v\|^2}{2}-c_{16}(\|\nabla u\|^2+\|\nabla^m v\|^2)-\frac{\|v_t\|^2}{2}.$$

(3.11)

$$\left|(g_2(u_1,v_1)-g_2(u_2,v_2),v_t)\right| \le \frac{c_{17}\|\nabla u\|}{2\lambda_1}+\frac{c_{18}\|\nabla^m v\|}{2\lambda_1^m}+\|v_t\|^2. \qquad (3.12)$$

Then,

$$\frac{1}{2}\frac{d}{dt}[\|v_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)\|\nabla^m v\|^2]$$
$$\le (\frac{3}{2}-\lambda_1^m\beta)\|v_t\|^2+(c_{16}+\frac{k_1}{2}+\frac{c_{18}}{2\lambda_1^m})\|\nabla^m v\|^2+(\frac{c_{17}}{2\lambda_1}+c_{16})\|\nabla u\|^2.$$

(3.13)

Plug (3.9) into (3.13), we have

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2+\|v_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(\|\nabla u\|^2+\|\nabla^m v\|^2)]$$
$$\le(\frac{3}{2}-\lambda_1\beta)\|u_t\|^2+(\frac{3}{2}-\lambda_1^m\beta)\|v_t\|^2+(c_{13}+c_{16}+\frac{k_1}{2}+\frac{c_{14}}{2\lambda_1}+\frac{c_{17}}{2\lambda_1})\|\nabla u\|^2+(c_{13}+c_{16}+\frac{k_1}{2}+\frac{c_{15}}{2\lambda_1^m}+\frac{c_{18}}{2\lambda_1^m})\|\nabla^m v\|^2.$$

(3.14)

Obviously, there exists $\varepsilon_3=\max\left\{c_{13}+c_{16}+\frac{k_1}{2}+\frac{c_{14}}{2\lambda_1}+\frac{c_{17}}{2\lambda_1},c_{13}+c_{16}+\frac{k_1}{2}+\frac{c_{15}}{2\lambda_1^m}+\frac{c_{18}}{2\lambda_1^m}\right\}$,

$\varepsilon_3\le cM(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)$,So, taking $\varepsilon_4=\max\left\{\frac{3}{2}-\lambda_1\beta,\frac{3}{2}-\lambda_1^m\beta,\varepsilon_3\right\}$, we get

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2+\|v_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(\|\nabla u\|^2+\|\nabla^m v\|^2)]$$
$$\le\varepsilon_4[\|u_t\|^2+\|v_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(\|\nabla u\|^2+\|\nabla^m v\|^2)]. \qquad (3.15)$$

By using Gronwall's inequality for (3.15), we have

$$\|u_t\|^2+\|v_t\|^2+M(\|\nabla u_2\|^2+\|\nabla^m v_2\|^2)(\|\nabla u\|^2+\|\nabla^m v\|^2)$$
$$\le[\|u_t(0)\|^2+\|v_t(0)\|^2+M(\|\nabla u_2(0)\|^2+\|\nabla^m v_2(0)\|^2)(\|\nabla u(0)\|^2+\|\nabla^m v(0)\|^2)]e^{\varepsilon_4 t}.$$

(3.16)

which imply that

$$w(x,t)=0. \qquad (3.17)$$

Therefore,

$$w_1=\begin{pmatrix}u_1\\v_1\end{pmatrix}=w_2=\begin{pmatrix}u_2\\v_2\end{pmatrix}. \qquad (3.18)$$

Since, we get the uniqueness of the solution.

3.2. The Global Attractor

Theorem3.2. [13] Let E be a Banach space, and $\{S(t)\}_{(t\ge0)}$ are the semigroup operator on E. $S(t+\tau)=S(t)S(\tau)(\forall t,\tau\ge0), S(0)=I$,where I is a unit operator. Set $S(t)$ satisfies the follow conditions:

1). $S(t)$ is uniformly bounded, namely $\forall R>0,\|u\|_E\le R$.It exists a constant $C(R)$,so that

$$\|S(t)u\|_E\le C(R). \qquad (t\in[0,+\infty)); \qquad (3.19)$$

2). It exists a bounded absorbing set $B_0\subset E$, namely, $\forall B\subset E$,it exists a constant t_0 ,so that

$$S(t)B\subset B_0(t\ge t_0); \qquad (3.20)$$

where B_0 and B are bounded sets.

3)When $t>0$, $S(t)$ is a completely continuous operator. Therefore, the semigroup operator $S(t)$ exists a compact global attractor A .

Theorem3.3. Under the Lemma2.5., Lemma 2.6. and Theorem3.1. equations have a global attractor

$$A = w(B_0) = \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau} S(t)B_0}, \tag{3.21}$$

where $B_0 = \left\{(u,v,p,q)\in E_1 : \|(u,v,p,q)\|_{E_1} = \|u\|_{H_0^2(\Omega)} + \|v\|_{H_0^{2m}(\Omega)} + \|p\|_{H_0^1(\Omega)} + \|q\|_{H_0^m(\Omega)} \le R_0 + R_1\right\}$.

B_0 is the bounded absorbing set of E_1 ,and satisfies

1) $S(t)A = A, t>0$.

2) $\lim_{t\to\infty} dist(S(t)B, A) = 0$, here $B\subset E_1$,and it is abounded set,

$$\lim_{t\to\infty} dist(S(t)B, A) = \sup_{x\in B}(\inf_{y\in A}\|S(t)x-y\|_{E_1}) \to 0, t\to\infty. \tag{3.22}$$

Proof. With the conditions of Theorem3.1., it exists the solution semigroup $S(t)$, $S(t): E_1 \to E_1$.

(1) From Lemma2.5. to Lemma2.6., we can obtain that $\forall B\subset E_1$ is a bounded set that includes in the ball $\left\{\|(u,v,p,q)\|_{E_1} \le R\right\}$.

$$\begin{aligned}\|S(t)(u_0,v_0,p_0,q_0)\|^2 &= \|u\|^2_{H_0^2(\Omega)} + \|v\|^2_{H_0^{2m}(\Omega)} + \|p\|^2_{H_0^1(\Omega)} + \|q\|^2_{H_0^m(\Omega)} \\ &\le \|u_0\|^2_{H_0^2(\Omega)} + \|v_0\|^2_{H_0^{2m}(\Omega)} + \|p_0\|^2_{H_0^1(\Omega)} + \|q_0\|^2_{H_0^m(\Omega)}. \\ &\le R^2 + C \quad (t\ge 0, (u_0,v_0,p_0,q_0)\in B)\end{aligned}$$

(3.23)

which imply that $S(t)(t\ge 0)$ is uniformly bounded in E_1.

(2) Furthermore, for any $\|(u_0,v_0,p_0,q_0)\|_{E_1}$,when $t\ge \max\{t_0,t_1\}$,we get

$$\|S(t)(u_0,v_0,p_0,q_0)\|^2 = \|u\|^2_{H_0^2(\Omega)} + \|v\|^2_{H_0^{2m}(\Omega)} + \|p\|^2_{H_0^1(\Omega)} + \|q\|^2_{H_0^m(\Omega)} \le R_0^2 + R_1^2.$$

(3.24)

So we get B_0 is the bounded absorbing set.

(3) Since $E_1 \to E_0$ is compact embedded, which means that the bounded set in E_1 is the compact set in E_0, so the semigroup operator $S(t)$ exists a compact global attractor A .

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