THE GLOBAL ATTRACTORS FOR A CLASS OF NONLINEAR COUPLED KIRCHHOFF-TYPE EQUATIONS

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ABSTRACT

This paper deals with the initial boundary value problem for a class of Kirchhoff-type coupled equations with strong damping and source terms. By using the Galerkin method, the existence and uniqueness of the solutions are got. And then we obtain the existence of the global attractor.

Keywords: Kirchhoff-type equations; The Global attractor; Galerkin method.

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1. INTRODUCTION

In this paper, we concerned with the existence of the global attractor for the following Kirchhoff-type equations:

$$u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u-\beta\Delta u_t+g_1(u,v)=f_1(x), \tag{1.1}$$

$$v_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta v-\beta\Delta v_t+g_2(u,v)=f_2(x), \tag{1.2}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \tag{1.3}$$

$$v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, \tag{1.4}$$

$$u|_{\partial\Omega}=v_{\partial\Omega}=0, \tag{1.5}$$

where Ω is a bounded domain in R^2 with the smooth boundary $\partial\Omega$, $\beta>0$ is a constant. $M(s)$ is a nonnegative C^1 function, $-\Delta u_t$ and $-\Delta v_t$ are strongly damping, $g_1(u,v)$ and $g_2(u,v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing function.

It is known that Kirchhoff (1883) [1] first investigated the following nonlinear vibration of an elastic string for $\delta=f=0$:

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}+f;\quad 0\le x\le L,t\ge 0 \tag{1.6}$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density , p_0 the initial axial tension , E the Young modulus, L the length, f the external force, δ the resistance modulus.

R Lou, P Lv, G Lin [2] considered a class of generalized nonlinear Kirchhoff-Sine-Gordon equation

$$u_{tt}-\beta\Delta u_t+\alpha u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+g(\sin u)=f(x), \tag{1.7}$$

$$u(x,t)=0,x\in\partial\Omega,t\ge 0, \tag{1.8}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_t(x), x\in\Omega, \tag{1.9}$$

where Ω is a bounded domain of $R^n(n\geq 1)$ with a smooth boundary $\partial\Omega$; α is the dissi- pation coefficient ; β is a positive constant ; and $f(x)$ is the external interference .then they prove the existence and uniqueness of solution to the initial value condition, they study the global attractors of the equation.

G Lin ,Y Gao [3] study the longtime behavior of solution to the initial boundary value problem for a class of strongly damped Higher-order Kirchhoff-type equations

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x),(x,t)\in\Omega\times[0,+\infty), \tag{1.10}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \tag{1.11}$$

$$u(x,t)=0, \frac{\partial^i u}{\partial v^i}=0, i=1,\ldots,m-1, x\in\partial\Omega, t\in[0,+\infty), \tag{1.12}$$

where $m>1$ is an integer constant , $\alpha>0, \beta>0$ are constants and q is a real number. Ω is a bounded domain of R^n with a smooth boundary $\partial\Omega$ and v is the unit outward norm- al on $\partial\Omega$. $g(u)$ is a nonlinear function specified later. Then they prove the existence

and uniqueness of the solution by the lemmas and the Galerkin method. They obtain to the existence of the global attractor in $H_0^m\times L^2(\Omega)$.

Shun-Tang Wu [4] consider the initial boundary value problem for the following non- linear wave equations of Kirchhoff type:

$$u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u+\int_0^t g(t-s)\Delta u(s)ds+|u_t|^{p-1}u_t=f_1(u,v) \text{ in } \Omega\times[0,\infty),$$

$$v_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta v+\int_0^t h(t-s)\Delta u(s)ds+|v_t|^{q-1}v_t=f_2(u,v) \text{ in } \Omega\times[0,\infty),$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \tag{1.13}$$

$$v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, \tag{1.14}$$

$$u(x,t)=v(x,t)=0, x\in\partial\Omega, t>0, \tag{1.15}$$

where Ω is a bounded domain in $R^n(n=1,2,3)$ with a smooth boundary $\partial\Omega$, $M(r)$ is a nonnegative C^1 function like $M(S)=M_0+\alpha s^\gamma$, with $M_0\geq 0$, $\alpha\geq 0$, $M_0+\alpha>0$ and $\gamma>0$,and $g,h:R^+\to R^+$, $f_i(\cdot,\cdot):R^2\to R, i=1,2$, are given functions which will be specified later. They obtain the blow-up of solutions in finite time when the initialenergy is nonnegative.

Yaojun Ye [5] studied the global existence and energy decay of solutions for coupled system of higher-order Kirchhoff-type equations with nonlinear dissipative and source terms in a bounded domain:

$$u_{tt}+\phi\left(\left\|D^{m_1}u\right\|^2+\left\|D^{m_2}v\right\|^2\right)(-\Delta)^{m_1}u+a|u_t|^{q-2}u_t=f_1(u,v), x\in\Omega, t>0, \tag{1.16}$$

$$v_{tt}+\phi\left(\left\|D^{m_1}u\right\|^2+\left\|D^{m_2}v\right\|^2\right)(-\Delta)^{m_2}v+a|v_t|^{q-2}v_t=f_2(u,v), x\in\Omega, t>0, \tag{1.17}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \tag{1.18}$$

$$v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, \tag{1.19}$$

$$\frac{\partial^i u}{\partial \upsilon^i}=0, i=0,1,2,\ldots,m_1-1, \tag{1.20}$$

$$\frac{\partial^j v}{\partial \upsilon^j}=0,\ j=0,1,2,\ldots,m_2-1, \tag{1.21}$$

They prove the existence of global solutions for this problem by constructing a stable set in $H_0^{m_1}(\Omega)\times H_0^{m_2}(\Omega)$ and give the decay estimate of global solutions by applying a lemma of V.Komornik.

2. Preliminaries

For brevity, we define

$$H=L^2(\Omega), V_1=H_0^1(\Omega), V_2=H^2(\Omega)\cap H_0^1(\Omega), \tag{2.1}$$

$$E_0=H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)=V_1\times V_1\times H\times H, \tag{2.2}$$

$$E_1=\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)\times H_0^1(\Omega)=V_2\times V_2\times V_1\times V_1. \tag{2.3}$$

In addition,

$$\|\cdot\|=\|\cdot\|_{L^2}, \|\cdot\|_{H_0^1}=\|\cdot\|_{H_0^1(\Omega)}, (u,v)=\int_\Omega u\cdot v dx. \tag{2.4}$$

Lemma2.1 (Young's inequality[6]) For any $\varepsilon>0$ and $a,b>0$,then

$$ab\le\frac{\varepsilon^p}{p}a^p+\frac{1}{q\varepsilon^q}b^q, \left(\frac{1}{p}+\frac{1}{q}=1, p>1, q>1\right). \tag{2.5}$$

Lemma2.2 (Hölder inequality[6]) Let $\frac{1}{p}+\frac{1}{q}=1, p>1, q>1$,for any $f(x)\in L^p(\Omega)$, $g(x)\in L^q(\Omega)$,then

$$\int_G |f(x)g(x)|dx\le\left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_G |g(x)|^q dx\right)^{\frac{1}{q}}. \tag{2.6}$$

Lemma2.3 (Poincare inequality[6]) If $\Omega\subset R^n$,then ,

$$\|u\|_{L^2(\Omega)}\le\lambda_1^{-\frac{1}{2}}\|\nabla u\|_{L^2(\Omega)}, \forall u\in H_0^1(\Omega), \tag{2.7}$$

where $\lambda_1>0$ is the first eigenvalue of $-\Delta$ in H_0^1.

Lemma2.4 (Gronwall's inequality[6]) If $\forall t\in[t_0,+\infty), y(t)\ge 0$ and $\frac{dy}{dt}+gy\le h$,such that

$$y(t)\le y(t_0)e^{-g(t-t_0)}+\frac{h}{g}, t\ge t_0, \tag{2.8}$$

where $g>0, h>0$ are constants.

Lemma2.5 (Gagliardo-Nirenberg inequality [6]) Let $\Omega\subset R^n, n\ge 2$, for any $m\ge 1, p\ge 1$, $u\in W_0^{1,m}(\Omega)$, then

$$\|u\|_{L^p(\Omega)}\le C\|\nabla u\|_{L^m(\Omega)}^{\alpha}\|u\|_{L^r(\Omega)}^{1-\alpha}, \tag{2.9}$$

where $\alpha=\left(\frac{1}{r}-\frac{1}{p}\right)\cdot\left(\frac{1}{r}-\frac{n-m}{nm}\right)^{-1}$, $C=C(\Omega)$ is a constant.

Next, we give some assumptions

(H_1)

$$J(u,v)=\int_\Omega\int_0^u g_1(\xi,v)d\xi dx+\int_\Omega\int_0^v g_2(u,\eta)d\eta dx>0. \tag{2.10}$$

There exists constant $\mu>0, C_1>0, C_2(\mu)>0$,such that

$$\left(g_1(u,v),u\right)+\left(g_2(u,v),v\right)\geq C_1J(u,v)-\varepsilon^2\mu\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\varepsilon C_2(\mu). \quad (2.11)$$

(H_2)

$$M(s)\in C^1\left(R^+\right). \quad (2.12)$$

(H_3)

There exists constant $\beta<\frac{3\lambda_1}{2},\beta_1,\beta_2$, such that

$$1<\beta\leq M(s)\leq\beta_1,\ \beta_2=\begin{cases}\beta,\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)>0\\ \beta_1,\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)<0\end{cases}, \quad (2.13)$$

and

$$M(s)\cdot s\geq\int_0^s M(t)dt>0. \quad (2.14)$$

(H_4)

There exist constant $C_3>0$, such that

$$\left|g_i(u,v)\right|\leq C_3\left(1+|u|^s+|v|^r\right),i=1,2,g_i(u,v)\in C^2, \quad (2.15)$$

where $2\leq s<\infty(n=2)$.

Lemma2.6 Assume (H_1)-(H_2)hold, and $(u_0,v_0,p_0,q_0)\in E_0$, $f_1\in L^2(\Omega)$, $f_2\in L^2(\Omega)$, $p=u_t+\varepsilon u$, $q=v_t+\varepsilon v$, then the solution (u,v,p,q) of the problem (1.1)-(1.5) satisfies $(u,v,p,q)\in L^\infty((0,+\infty);E_0)$,

and

$$\|p\|^2+\|q\|^2+\|\nabla u\|^2+\|\nabla v\|^2\leq y(0)e^{-C_4t}+\frac{C_5}{C_4}, \quad (2.16)$$

where

$$y(0)=\|u_1+\varepsilon u_0\|^2+\|v_1+\varepsilon v_0\|^2+\int_0^{\|\nabla u_0\|^2+\|\nabla v_0\|^2}M(s)ds+2J(u_0,v_0), \quad (2.17)$$

$$C_4=\min\left\{\frac{\varepsilon}{2},\lambda_1\beta-2\varepsilon^2-2\varepsilon,2C_1\right\}, (2.18)$$

$$C_5=\frac{\|f_1\|^2+\|f_2\|^2}{\varepsilon^2}+2\varepsilon C_2(\mu). \quad (2.19)$$

so there exists R_0 and $t_0=t_0(\Omega)>0$, such that

$$\|p\|^2+\|q\|^2+\|\nabla u\|^2+\|\nabla v\|^2\leq R_0^{\ 2},t>t_0. \quad (2.20)$$

Proof.

Taking the scalar product in $L^2(\Omega)$ of equation (1.1) with $p=u_t+\varepsilon u$, and we obtain

$$\left(u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u-\beta\Delta u_t+g_1(u,v),p\right)=\left(f_1(x),p\right), \quad (2.21)$$

where $0<\varepsilon<\frac{3\beta}{2}\cdot\left(\frac{1}{\lambda_1}+\beta+2\mu\right)$.

Then, by using Lemma2.1, Lemma2.2, we have

$$(u_{tt},p)\geq\frac{1}{2}\frac{d}{dt}\|p\|^2-\frac{\varepsilon^2+2\varepsilon}{2}\|p\|^2-\frac{\varepsilon^2}{2}\|u\|^2. \tag{2.22}$$

$$\begin{aligned}&\left(-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u,p\right)\\&=\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\|\nabla u\|^2+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\nabla u\|^2\end{aligned}. \tag{2.23}$$

$$(-\beta\Delta u_t,p)\geq\frac{\beta}{2}\|\nabla p\|^2-\frac{\varepsilon^2\beta}{2}\|\nabla u\|^2. \tag{2.24}$$

$$(g_1(u,v),p)=\frac{d}{dt}\int_\Omega\int_0^u g_1(\xi,v)d\xi dx+(g_1(u,v),\varepsilon u). \tag{2.25}$$

$$(f_1,p)\leq\frac{1}{2\varepsilon^2}\|f_1\|^2+\frac{\varepsilon^2}{2}\|p\|^2. \tag{2.26}$$

So, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|p\|^2-\frac{\varepsilon^2+2\varepsilon}{2}\|p\|^2-\frac{\varepsilon^2}{2}\|u\|^2+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\|\nabla u\|^2\\&+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\nabla u\|^2+\frac{\beta}{2}\|\nabla p\|^2-\frac{\varepsilon^2\beta}{2}\|\nabla u\|^2\\&+\frac{d}{dt}\int_\Omega\int_0^u g_1(\xi,v)d\xi dx+(g_1(u,v),\varepsilon u)\\&\leq\frac{1}{2\varepsilon^2}\|f_1\|^2+\frac{\varepsilon^2}{2}\|p\|^2\end{aligned}. \tag{2.27}$$

Next, taking the scalar product in $L^2(\Omega)$ of equation (1.2) with $q=v_t+\varepsilon v$, and then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|q\|^2-\frac{\varepsilon^2+2\varepsilon}{2}\|q\|^2-\frac{\varepsilon^2}{2}\|v\|^2+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\|\nabla v\|^2\\&+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\nabla v\|^2+\frac{\beta}{2}\|\nabla q\|^2-\frac{\varepsilon^2\beta}{2}\|\nabla v\|^2\\&+\frac{d}{dt}\int_\Omega\int_0^v g_1(u,\eta)d\eta dx+(g_2(u,v),\varepsilon v)\\&\leq\frac{1}{2\varepsilon^2}\|f_2\|^2+\frac{\varepsilon^2}{2}\|q\|^2\end{aligned}. \tag{2.28}$$

Adding (2.26) to (2.27), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|p\|^2+\|q\|^2\right)-\frac{\varepsilon^2+2\varepsilon}{2}\left(\|p\|^2+\|q\|^2\right)-\frac{\varepsilon^2}{2}\left(\|u\|^2+\|v\|^2\right)-\frac{\varepsilon^2\beta}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\&+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\&+\frac{\beta}{2}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)+\frac{d}{dt}\int_\Omega\int_0^u g_1(\xi,v)d\xi dx+\frac{d}{dt}\int_\Omega\int_0^v g_2(u,\eta)d\eta dx\\&+(g_1(u,v),\varepsilon u)+(g_2(u,v),\varepsilon v)\leq\frac{1}{2\varepsilon^2}\|f_1\|^2+\frac{\varepsilon^2}{2}\|p\|^2+\frac{1}{2\varepsilon^2}\|f_2\|^2+\frac{\varepsilon^2}{2}\|q\|^2\end{aligned}, \tag{2.29}$$

where

$$\frac{d}{dt}\int_0^{\|\nabla u\|^2+\|\nabla v\|^2} M(s)ds = M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla v\|^2\right). \qquad (2.30)$$

By using Lemma2.3, we have

$$-\frac{\varepsilon^2}{2}\left(\|u\|^2+\|v\|^2\right) \geq -\frac{\varepsilon^2}{2\lambda_1}\left(\|\nabla u\|^2+\|\nabla v\|^2\right), \qquad (2.31)$$

so, we can get

$$\begin{aligned} &2\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-2\left(\frac{\varepsilon^2}{2\lambda_1}+\frac{\varepsilon^2\beta}{2}+\varepsilon^2\mu\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &> \frac{\varepsilon}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \qquad , \qquad (2.32)\\ &\geq \frac{\varepsilon}{2}\int_0^{\|\nabla u\|^2+\|\nabla v\|^2} M(s)ds \end{aligned}$$

where $\varepsilon < \frac{3\beta}{2}\cdot\left(\frac{1}{\lambda_1}+\beta+2\mu\right)$.

By using lemma2.3, we have

$$\frac{d}{dt}\left[\int_0^{\|\nabla u\|^2+\|\nabla v\|^2} M(s)ds+\|p\|^2+\|q\|^2+2J(u,v)\right]+C_4\left(\int_0^{\|\nabla u\|^2+\|\nabla v\|^2} M(s)ds+\|p\|^2+\|q\|^2+2J(u,v)\right)\leq C_5$$

where

$$C_4=\min\left\{\frac{\varepsilon}{2},\lambda_1\beta-2\varepsilon^2-2\varepsilon,2C_1\right\}, C_5=\frac{\|f_1\|^2+\|f_2\|^2}{\varepsilon^2}+2\varepsilon C_2(\mu).$$

We know

$y(t)=\int_0^{\|\nabla u\|^2+\|\nabla v\|^2} M(s)ds+\|p\|^2+\|q\|^2+2J(u,v)\geq 0$, then by using lemma 2.4, we get

$$y(t)\leq y(0)e^{-C_4 t}+\frac{C_5}{C_4}. \qquad (2.33)$$

Next

$$\overline{\lim_{x\to\infty}}\|(u,v,p,q)\|_{E_0}^2=\|\nabla u\|^2+\|\nabla v\|^2+\|p\|^2+\|q\|^2\leq\frac{C_5}{C_4}, \qquad (2.34)$$

so there exists R_0 and $t_0=t_0(\Omega)>0$,such that

$$\|(u,v,p,q)\|_{E_0}^2=\|p\|^2+\|q\|^2+\|\nabla u\|^2+\|\nabla v\|^2\leq R_0{}^2, t>t_0. \qquad (2.35)$$

Lemma2.7 In addition to the assumptions of lemma2.6, If (H_6):$(u_0,v_0,p_0,q_0)\in E_1$, $f_1\in H_0^1(\Omega)$, $f_2\in H_0^1(\Omega)$, then the solution (u,v,p,q) of the problem (1.1)-(1.5) satisf- ies $(u,v,p,q)\in L^\infty((0,+\infty);E_1)$, and

$$\|\nabla p\|^2+\|\nabla q\|^2+\|\Delta u\|^2+\|\Delta v\|^2\leq y(0)e^{-C_{10}t}+\frac{2C_9}{C_{10}}, \qquad (2.36)$$

where

$$y(0)=\|\nabla(u_1+\varepsilon u_0)\|^2+\|\nabla(v_1+\varepsilon v_0)\|^2+M\left(\|\nabla u_0\|^2+\|\nabla v_0\|^2\right)\left(\|\Delta u_0\|^2+\|\Delta v_0\|^2\right),$$

$$C_{10}=\min\left\{2\left(\frac{\lambda_1(\beta-\varepsilon)}{2}-\varepsilon^2-\varepsilon\right),\frac{\varepsilon\beta}{8\beta_2}\right\},C_9=\frac{\|\nabla f_1\|^2+\|\nabla f_2\|^2}{\varepsilon^2}+C_8,$$

so there exists R_1 and $t_1=t_1(\Omega)>0$, such that

$$\|\nabla p\|^2+\|\nabla q\|^2+\|\Delta u\|^2+\|\Delta v\|^2\le R_1^{\,2},t>t_1. \tag{2.37}$$

Proof.

Taking the scalar product in $L^2(\Omega)$ of equation(1.1) with $(-\Delta)p=(-\Delta)u_t+\varepsilon(-\Delta)u$, $0<\varepsilon<\min\left\{\beta,\frac{3\lambda_1\beta}{2(1+\beta\lambda_1)}\right\}$, we obtain

$$\left(u_{tt}-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u-\beta\Delta u_t+g_1(u,v),(-\Delta)p\right)=\left(f_1(x),(-\Delta)p\right), \tag{2.38}$$

and by using Lemma2.1-Lemma2.2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla p\|^2-\frac{\varepsilon^2+2\varepsilon}{2}\|\nabla p\|^2-\frac{\varepsilon^2}{2}\|\nabla u\|^2\\&+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\|\Delta u\|^2+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\Delta u\|^2\\&+\frac{\beta}{2}\|\Delta p\|^2-\frac{\varepsilon^2\beta}{2}\|\Delta u\|^2-\frac{1}{2\varepsilon}\|g_1(u,v)\|^2-\frac{\varepsilon}{2}\|\Delta p\|^2\\&\le\frac{1}{2\varepsilon^2}\|\nabla f_1\|^2+\frac{\varepsilon^2}{2}\|\nabla p\|^2\end{aligned}. \tag{2.39}$$

Taking the scalar product in $L^2(\Omega)$ of equation (1.2) with $(-\Delta)q=(-\Delta)v_t+\varepsilon(-\Delta)v$, $0<\varepsilon<\min\left\{\beta,\frac{3\lambda_1\beta}{2(1+\beta\lambda_1)}\right\}$, and by repeating these steps, then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla q\|^2-\frac{\varepsilon^2+2\varepsilon}{2}\|\nabla q\|^2-\frac{\varepsilon^2}{2}\|\nabla v\|^2\\&+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\|\Delta v\|^2+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\|\Delta v\|^2\\&+\frac{\beta}{2}\|\Delta q\|^2-\frac{\varepsilon^2\beta}{2}\|\Delta v\|^2-\frac{1}{2\varepsilon}\|g_2(u,v)\|^2-\frac{\varepsilon}{2}\|\Delta q\|^2\\&\le\frac{1}{2\varepsilon^2}\|\nabla f_2\|^2+\frac{\varepsilon^2}{2}\|\nabla q\|^2\end{aligned}. \tag{2.40}$$

Adding (2.38) to (2.39), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)-\frac{\varepsilon^2+2\varepsilon}{2}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)-\frac{\varepsilon^2}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\&+\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\\&+\frac{\beta}{2}\left(\|\Delta p\|^2+\|\Delta q\|^2\right)-\frac{\varepsilon^2\beta}{2}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)-\frac{1}{2\varepsilon}\left(\|g_1(u,v)\|^2+\|g_2(u,v)\|^2\right)\\&-\frac{\varepsilon}{2}\left(\|\Delta p\|^2+\|\Delta q\|^2\right)\le\frac{1}{2\varepsilon^2}\|\nabla f_1\|^2+\frac{\varepsilon^2}{2}\|\nabla p\|^2+\frac{1}{2\varepsilon^2}\|\nabla f_2\|^2+\frac{\varepsilon^2}{2}\|\nabla q\|^2\end{aligned}. \tag{2.41}$$

Therefore, by (H_4), we can get

$$\frac{1}{2}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\geq \beta_2\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right). \qquad (2.42)$$

By using lemma2.3, (H_2), (H_3), we have

$$\begin{aligned}&-\frac{\varepsilon^2+2\varepsilon}{2}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)+\frac{\beta}{2}\left(\|\Delta p\|^2+\|\Delta q\|^2\right)\\&-\frac{\varepsilon}{2}\left(\|\Delta p\|^2+\|\Delta q\|^2\right)-\frac{\varepsilon^2}{2}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)\\&=-\left(\varepsilon^2+\varepsilon\right)\left(\|\nabla p\|^2+\|\nabla q\|^2\right)+\frac{\beta-\varepsilon}{2}\left(\|\Delta p\|^2+\|\Delta q\|^2\right)\\&\geq\left(\frac{\lambda_1(\beta-\varepsilon)}{2}-\varepsilon^2-\varepsilon\right)\left(\|\nabla p\|^2+\|\nabla q\|^2\right)\end{aligned}, \qquad (2.43)$$

where $\varepsilon<\beta$,

and

$$\begin{aligned}&-\frac{\varepsilon^2}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\Delta u\|^2+\|\Delta v\|^2\right)-\frac{\varepsilon^2\beta}{2}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\\&\geq-\left(\frac{\varepsilon^2}{2\lambda_1}+\frac{\varepsilon^2\beta}{2}\right)\left(\|\Delta u\|^2+\|\Delta v\|^2\right)+\frac{\varepsilon}{4}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\\&+\frac{3\varepsilon}{4}M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\geq\frac{\varepsilon\beta}{4}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\end{aligned} \qquad (2.44)$$

where $\varepsilon<\dfrac{3\lambda_1\beta}{2(1+\beta\lambda_1)}$.

By using (H_5), we have

$$\|g_i(u,v)\|^2=\int_\Omega|g_i(u,v)|^2dx\leq\int_\Omega C_3\left(1+|u|^s+|v|^r\right)^2dx\leq C_3^{\,2}\left(1+\|u\|_{L^{2s}}^{2s}+\|v\|_{L^{2r}}^{2r}\right). \qquad (2.45)$$

So, by the Lemma2.5, we have

$$\|u\|_{L^{2s}}^{2s}\leq C_6\|\nabla u\|^{2s-2}\|u\|^2 \quad \text{for } 1\leq s\leq\infty(n=2), \qquad (2.46)$$

$$\|u\|_{L^{2r}}^{2r}\leq C_7\|\nabla u\|^{2r-2}\|u\|^2 \quad \text{for } 1\leq r\leq\infty(n=2). \qquad (2.47)$$

Therefore

$$\frac{1}{2\varepsilon}\left(\|g_1(u,v)\|^2+\|g_2(u,v)\|^2\right)\leq C_8. \qquad (2.48)$$

So, we can get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla p\|^2+\|\nabla q\|^2\right)+\beta_2\frac{d}{dt}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\\&+\left(\frac{\lambda_1(\beta-\varepsilon)}{2}-\varepsilon^2-\varepsilon\right)\left(\|\nabla p\|^2+\|\nabla q\|^2\right)+\frac{\varepsilon\beta}{4}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\leq C_9\end{aligned}, \qquad (2.49)$$

then,

$$\frac{d}{dt}\left[\|\nabla p\|^2+\|\nabla q\|^2+2\beta_2\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\right]+C_{10}\left(\|\nabla p\|^2+\|\nabla q\|^2+2\beta_2\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\right)\leq 2C_9$$

where

$$C_{10}=\min\left\{2\left(\frac{\lambda_1(\beta-\varepsilon)}{2}-\varepsilon^2-\varepsilon\right),\frac{\varepsilon\beta}{8\beta_2}\right\},C_9=\frac{\|\nabla f_1\|^2+\|\nabla f_2\|^2}{\varepsilon^2}+C_8.$$

We know

$$y(t)=\|\nabla p\|^2+\|\nabla q\|^2+2\beta_2\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\geq 0, \tag{2.50}$$

then by using lemma 2.4, we get

$$y(t)\leq y(0)e^{-C_{10}t}+\frac{2C_9}{C_{10}}, \tag{2.51}$$

next

$$\overline{\lim_{x\to\infty}}\|(u,v,p,q)\|_{E_1}^2=\|\Delta u\|^2+\|\Delta v\|^2+\|\nabla p\|^2+\|\nabla q\|^2\leq\frac{2C_9}{C_{10}}. \tag{2.52}$$

So there exists R_1 and $t_1=t_1(\Omega)>0$, such that

$$\|(u,v,p,q)\|_{E_1}^2=\|\nabla p\|^2+\|\nabla q\|^2+\|\Delta u\|^2+\|\Delta v\|^2\leq R_1^2, t>t_1. \tag{2.53}$$

3.Global Attractor.

3.1 The Existence and Uniqueness of Solution

Theorem3.1 Assume (H_2), (H_3), (H_5) hold, and $(u_0,v_0,p_0,q_0)\in E_1$, $f_1\in H_0^1(\Omega)$, $f_2 \in H_0^1(\Omega)$, hence, Equation (1.1)-(1.5)exists a unique smooth solution $(u,v,p,q)\in L^\infty((0,+\infty);E_1)$.

Proof. By the Galerkin method, Lemma2.6, Lemma2.7, we can obtain the existence of solutions. Next, we prove the uniqueness of solution in detail.

Suppose $\omega_1=\begin{pmatrix}u_1\\v_1\end{pmatrix}$, $\omega_2=\begin{pmatrix}u_2\\v_2\end{pmatrix}$ are two solutions of the problems .

Let $\omega=\begin{pmatrix}u\\v\end{pmatrix}=\omega_1-\omega_2$, then we have,

$$\begin{cases}u_{tt}-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta u_1+M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta u_2-\beta\Delta u_t+g_1(u_1,v_1)-g_1(u_2,v_2)=0\\ v_{tt}-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta v_1+M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta v_2-\beta\Delta v_t+g_2(u_1,v_1)-g_2(u_2,v_2)=0\end{cases}$$

By using u_t to inner product of

$$u_{tt}-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta u_1+M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta u_2-\beta\Delta u_t+g_1(u_1,v_1)-g_1(u_2,v_2)=0$$

then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\left(M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta u_2-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta u_1,u_t\right)\\&+\beta\|\nabla u_t\|^2+\left(g_1(u_1,v_1)-g_1(u_2,v_2),u_t\right)=0\end{aligned}. \tag{3.1}$$

Then, by using Lagrange's mean value theorem, lemma2.5, lemma2.6, we have

$$\left(M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta u_2-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta u_1,u_t\right)$$
$$=\frac{1}{2}M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\frac{d}{dt}\|\nabla u\|^2+\left(M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\Delta u_1-M\left(\|\nabla u_1\|^2+\|\nabla v_1\|^2\right)\Delta u_1,u_t\right)$$
$$\geq\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\|\nabla u\|^2\right]-\frac{\|\nabla u\|^2}{2}\frac{d}{dt}M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)-C_{11}\left(\|\nabla u\|+\|\nabla v\|\right)\|u_t\|$$
$$\geq\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\|\nabla u\|^2\right]-\frac{\|\nabla u\|^2}{2}C_{11}-C_{12}\left(\|\nabla u\|+\|\nabla v\|\right)\|u_t\|$$
$$\geq\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\|\nabla u\|^2\right]-\frac{\|\nabla u\|^2}{2}C_{11}-C_{13}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\frac{\|u_t\|^2}{2}$$

and

$$\left|\left(g_1(u_1,v_1)-g_1(u_2,v_2),u_t\right)\right|=\left|\left(g_1(u_1,v_1)-g_1(u_1,v_2)+g_1(u_1,v_2)-g_1(u_2,v_2),u_t\right)\right|$$
$$=\left|\left(g_{1u}(\xi_1,v_1)(u_1-u_2)-g_{1v}(u_2,\xi_2)(v_1-v_2),u_t\right)\right| \tag{3.2}$$
$$\leq C_{14}\|u\|\cdot\|u_t\|+C_{15}\|v\|\cdot\|u_t\|\leq\frac{C_{14}\lambda_1^{-1}\|\nabla u\|^2}{2}+\frac{C_{15}\lambda_1^{-1}\|\nabla v\|^2}{2}+\|u_t\|^2$$

Next, we have

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\|\nabla u\|^2\right]$$
$$\leq\left(\frac{3}{2}-\lambda_1\beta\right)\|u_t\|^2+\left(\frac{C_{11}}{2}+C_{13}+\frac{C_{14}\lambda_1^{-1}}{2}\right)\|\nabla u\|^2+\left(\frac{C_{15}\lambda_1^{-1}}{2}+C_{13}\right)\|\nabla v\|^2. \tag{3.3}$$

Similar obtain

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2+\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\|\nabla v\|^2\right]$$
$$\leq\left(\frac{3}{2}-\lambda_1\beta\right)\|v_t\|^2+\left(\frac{C_{11}}{2}+C_{16}+\frac{C_{17}\lambda_1^{-1}}{2}\right)\|\nabla v\|^2+\left(\frac{C_{18}\lambda_1^{-1}}{2}+C_{16}\right)\|\nabla u\|^2. \tag{3.4}$$

Adding (3.3) to (3.4), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\frac{d}{dt}\left[M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\right]$$
$$\leq\left(\frac{3}{2}-\lambda_1\beta\right)\left(\|u_t\|^2+\|\nabla_t\|^2\right)+\left(\frac{C_{11}}{2}+C_{13}+\frac{C_{14}\lambda_1^{-1}}{2}\right)\|\nabla u\|^2 \quad . \tag{3.5}$$
$$+\left(\frac{C_{11}}{2}+C_{16}+\frac{C_{17}\lambda_1^{-1}}{2}\right)\|\nabla v\|^2+\left(\frac{C_{15}\lambda_1^{-1}}{2}+C_{13}\right)\|\nabla v\|^2+\left(\frac{C_{18}\lambda_1^{-1}}{2}+C_{16}\right)\|\nabla u\|^2$$

There exist $C_{19}=\max\left\{\frac{C_{11}}{2}+C_{13}+\frac{C_{14}\lambda_1^{-1}}{2}+\frac{C_{18}\lambda_1^{-1}}{2}+C_{16},\frac{C_{11}}{2}+C_{16}+\frac{C_{17}\lambda_1^{-1}}{2}+\frac{C_{15}\lambda_1^{-1}}{2}+C_{13}\right\}$,

such that

$$\left(\frac{C_{11}}{2}+C_{13}+\frac{C_{14}\lambda_1^{-1}}{2}\right)\|\nabla u\|^2+\left(\frac{C_{11}}{2}+C_{16}+\frac{C_{17}\lambda_1^{-1}}{2}\right)\|\nabla v\|^2$$
$$+\left(\frac{C_{15}\lambda_1^{-1}}{2}+C_{13}\right)\|\nabla v\|^2+\left(\frac{C_{18}\lambda_1^{-1}}{2}+C_{16}\right)\|\nabla u\|^2 \quad , \tag{3.6}$$
$$\le C_{19}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\le C_{19}M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)$$

and there exist

$$C_{20}=\max\left\{\frac{3}{2}-\lambda_1\beta, C_{19}\right\}, \tag{3.7}$$

such that

$$\frac{d}{dt}\tau(t)\le 2C_{20}\tau(t), \tag{3.8}$$

where

$$\tau(t)=\|u_t\|^2+\|v_t\|^2+M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right). \tag{3.9}$$

Therefore, we have

$$\tau(t)\le\tau(0)e^{2C_{20}t}, \tag{3.10}$$

where $\tau(0)=\|u_t(0)\|^2+\|v_t(0)\|^2+M\left(\|\nabla u_2(0)\|^2+\|\nabla v_2(0)\|^2\right)\left(\|\nabla u(0)\|^2+\|\nabla v(0)\|^2\right)=0$,

so

$$\|u_t\|^2+\|v_t\|^2+M\left(\|\nabla u_2\|^2+\|\nabla v_2\|^2\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\le 0. \tag{3.11}$$

Thus $\omega_1=\omega_2$, proof finished.

3.2. The Global Attractor.

Theorem3.2 [6] Let E is a Banach space ,and $\{S(t)\}(t\ge 0)$ are the semigroup operator on E, since $S(t):E\to E$, $S(t+s)=S(t)\cdot S(s)(\forall t,s\ge 0)$, $S(0)=I$, where I is a unit operator, set $S(t)$ satisfies the follow conditions:

1). $S(t)$ is uniformly bounded, namely $\forall R>0$, $\|u\|_E\le C(R)$, it exists a constant $C(R)$ so that

$$\|S(t)u\|_E\le C(R),\ \forall t\in[0,\infty); \tag{3.12}$$

2). It exists a bounded absorbing set $B_0\subset E$, namely, $B\subset E$, it exists a constant t_0 ,so that

$$S(t)B\subset B_0,(t\ge t_0); \tag{3.13}$$

where B_0 and B are bounded sets.

3). When $t>0$, $S(t)$ is a completely continuous operator. Therefore, the semigroup op- erator $S(t)$ exists a compact global attractor A.

Theorem3.3 [6] Under the assume of Lemma2.6, Lemma2.7, and Theorem3.1, equations (1.1)-(1.5) have a global attractor

$$\mathrm{A}=\omega(B_0)=\bigcap_{\sigma\ge 0}\overline{\bigcup_{t\ge\sigma}S(t)B_0}, \tag{3.14}$$

where $B_0=\left\{(u,v,p,q)\in E_1:\|(u,v,p,q)\|_{E_1}=\|u\|_{V_2}+\|v\|_{V_2}+\|p\|_{V_1}+\|q\|_{V_1}\le R_0+R_1\right\}$, B_0 is the bounded absorbing set of E_1, and satisfies

1) $S(t)\mathrm{A}=\mathrm{A}, t>0$;

2) $\lim_{t\to\infty} dis\left(S(t)B, \mathrm{A}\right) = 0$, where $B \subset E_1$ is a bounded set, and satisfies

$$dis\left(S(t)B, \mathrm{A}\right) = \sup_{x\in B}\left(\inf_{y\in \mathrm{A}} \left\|S(t)x - y\right\|_{E_1}\right) \to 0,\ t\to\infty. \qquad (3.15)$$

Proof:

With the conditions of Theorem3.1, it exists the solution semigroup $S(t)$, and $X = E_1$ $S(t): X \to X$.

(1) Form Lemma2.6 and Lemma2.7, we can obtain that $\forall B \subset X$ is a bounded set that in the ball $\left\{\|u, v, p, q\|_X \le R\right\}$,and

$$\left\|S(t)(u_0, v_0, p_0, q_0)\right\|_X^2 = \|u\|_{V_2} + \|v\|_{V_2} + \|p\|_{V_1} + \|q\|_{V_1} \le \|u_0\|_{V_2}^2 + \|v_0\|_{V_2}^2 + \|p_0\|_{V_1}^2 + \|q_0\|_{V_1}^2$$
$$\le 2R^2 + C_{21}, \quad \left(t \ge 0, (u_0, v_0, p_0, q_0) \in B\right)$$

which imply that $S(t)(t \ge 0)$ is uniformly bounded in X .

(2) Furthermore, for any $(u_0, v_0, p_0, q_0) \in E_1$, when $t \ge \max(t_1, t_2)$, we have

$$\left\|S(t)(u_0, v_0, p_0, q_0)\right\|_{E_1}^2 = \|u\|_{V_2}^2 + \|v\|_{V_2}^2 + \|p\|_{V_1}^2 + \|q\|_{V_1}^2 \le 2R_0^{\ 2} + 2R_1^{\ 2}, \qquad (3.16)$$

(3) Since $E_1 \to E_0$ is the compact embedded, which means that the bounded set in E_1 is is the compact set in E_0, so the semigroup operator $S(t)$ exists a compact global attractor A .

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