COMMON FIXED POINT THEOREMS FOR MAPS SATISFYING $\varphi-\phi$ CONTRACTIONS IN COMPLEX VALUED METRIC SPACES

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ABSTRACT

In this paper, we use a new method to generalize the conclusion in [Fixed Point Theory & Applications, 2018, 2018(1):1] into the complex valued metric spaces. A class of 5-dimensional functions ϕ was defined, and we get the conclusion that the family of maps satisfying ϕ contractions have a unique common fixed point in complex valued metric spaces by generalizing Cauchy principle. The conclusion we obtained generalize many fixed point results in complex valued metric spaces.

Key words: Complex valued metric spaces, Cauchy sequence, $\varphi-$ contractions, Fixed point

I Introduction

The fixed point theory has been developed and applicated in many areas since the principle of Banach contraction was introduced in 1922.[1] In recent years, the fixed point theory has been generalized and improved mostly in the 2 aspects, one is the concept of metric spaces, such as the 2-metric spaces which has 3 variable,[2] and the b-metric spaces which has the parameter 's' in the rectangle inequality. The other is the improvement of maps, the most classical one is the $\varphi-$ contraction. Meanwhile, some scholars introduced the concept of quasi-contraction, expansive maps. All these generalization has developed the fixed point theory deeply.

Recently, the author in [3] built the conception of complex valued metric spaces by defined an partially ordered relation in the set of complex number. This space obviously generalized the classical metric spaces. And they got the fixed point theorem for maps satisfying a kind of contraction condition by constructing sequences. Then, the paper [4][5][6] generalized the conclusion of [3].

The aim of this paper is to introduce a contractive function, and generalize the conclusion of [7] into complex valued metric spaces, and get some useful corollaries. Meanwhile, a new class of 5-dimensional function ϕ was defined to prove a family of maps under this function has an unique common fixed point. The conclusion we get generalize many results in the complex valued metric spaces.

Let £ be the set of complex number， $z_1, z_2 \in$ £ . The partially ordered relation is defined as:

$$z_1 \;\hat{}\; z_2 \Leftrightarrow (Re(z_1) \le Re(z_2)) \wedge (Im(z_1) \le Im(z_2)).$$

Which means when one of the following conditions hold, $z_1 \;\hat{}\; z_2$:

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$$(C_1)\ Re(z_1)=Re(z_2)\wedge Im(z_1)=Im(z_2);\quad (C_2)\ Re(z_1)<Re(z_2)\wedge Im(z_1)=Im(z_2);$$
$$(C_3)\ Re(z_1)=Re(z_2)\wedge Im(z_1)<Im(z_2);\quad (C_4)\ Re(z_1)<Re(z_2)\wedge Im(z_1)<Im(z_2).$$

Especially, when $z_1 \neq z_2$, and one of $(C_2),(C_3),(C_4)$ holds, we mark $z_1 \lnsim z_2$; if (C_4), then $z_1 < z_2$.

Obviously, we have the following conclusion:

(i) if $a,b \geq 0$ and $a \leq b$, then for each $z \in \mathbb{C}$ and $0 \precsim z, az \precsim bz$;

(ii) if $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$;

(iii) if $z_1 \precsim z_2, z_2 < z_3$, then $z_1 < z_3$;

Definition 1.1[3] Let X be a nonempty set. If map $d: X \times X \to \mathbb{C}$ satisfying the condition:

(i) For all $x,y \in X, 0 \precsim d(x,y)$ and $d(x,y)=0$ if and only if $x=y$;

(ii) For all $x,y \in X, d(x,y)=d(y,x)$;

(iii) For all $x,y,z \in X, d(x,y) \precsim d(x,z)+d(y,z)$;

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space.

Example 1.1[3] Let $X=\mathbb{C}$, we define $d: X\times X \to \mathbb{C}$ as:

$$d(z_1,z_2)=e^{ik}|z_1-z_2|, \text{ for all } z_1,z_2 \in X$$

and $k \in \mathbb{R}$, then (X,d) is a complex valued metric space.

Example 1.2[4] Let $X=\{a,b,c\}$. Defined $d: X\times X \to \mathbb{C}$ as:

$$d(a,a)=d(b,b)=d(c,c)=0,\ d(a,b)=d(b,a)=2+3\mathrm{i},$$
$$d(a,c)=d(c,a)=3+4\mathrm{i},\quad d(b,c)=d(c,b)=4+5\mathrm{i}.$$

Obviously, (X,d) is a complex valued metric spaces.

Definition 1.2[3] Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.

(i) If for every $c \in \mathbb{C}$ with $0<c$, there is $n_0 \in \mathbb{N}$ such that for all $n>n_0$, $d(x_n,x)<c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to $x \in X$ and x is the limit point of $\{x_n\}$. We denote this by $x_n \to x\ (n\to\infty)$ or $\lim_{n\to\infty} x_n = x$.

(ii) If for every $c \in \mathbb{C}$ with $0<c$, there is $n_0 \in \mathbb{N}$ such that for all $m>n>n_0$, $d(x_n,x_m)<c$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then X is said to be a complex valued metric space.

Definition 1.3[7] Let (X,d) be a complex valued metric space，map $T: X \to X$

$\{x,Tx,T^2x,\ldots\}$ is said to be the orbit of T in x, $\{x,Tx,T^2x,\ldots y,Ty,T^2y,\ldots\}$ is said to be the orbit of T in (x,y).

Throughout this paper, we define $D_T(x,x)$ and $D_T(x,y)$ by:

$$D_T(x,x) = \sup\{|d(u,v)| : u,v \in \{x,Tx,\ldots\}\}$$

$$D_T(x,y) = \sup\{|d(u,v)| : u,v \in \{x,Tx,\ldots y,Ty,\ldots\}\}$$

for any $x,y \in X$. That is, $D_T(x,x)$ is the diameter of the orbit $\{x,Tx,T^2x,\ldots\}$ of x.

Remark 1: This definition is a generalization of metric space [1] into complex valued metric space.

Definition 1.4[7] Let map $\varphi:[0,+\infty)\to[0,+\infty)$, where φ is continuous and for every $t>0, \varphi(t)<t$ holds. Meanwhile, for each $\varepsilon>0$, there is $\delta>0$, such that when $\varepsilon<t<\varepsilon+\delta$, $\varphi(t)\le\varepsilon$.

Without loss of generality, let $\varphi(0)=0$.

Definition 1.5 Defined $C_+ = \{a+bi \mid a\ge 0, b\ge 0\}$

And a nonempty ψ is composed of function ϕ ,in which $\phi:(C_+)^5 \to C_+$, and:

(i) ϕ is continuous about every variable of C_+^5 and monotonically increasing for the fourth and fifth variables.

(ii) There is $k \in £$, $|k|<1$, if $y ˆ\ \phi(x,x,y,x+y,y)$ or $y ˆ\ \phi(x,y,x,x+y,y)$, then $y ˆ\ kx$.

The example of $\phi\in\psi$ are as follows:

Example 1.3 Defined the function: $\phi:(C_+)^5 \to C_+$

$$\phi(t_1,t_2,t_3,t_4,t_5) = k_1t_1 + k_2t_2 + k_3t_3 + k_4t_4 + k_5t_5$$

in which $k_i \in ¡$ $, i = 1\ldots5$, and $\sum_{i=1}^{5} k_i < 1-k_4$, let $k = \max\{\frac{k_1+k_2+k_4}{1-k_3-k_4-k_5}, \frac{k_1+k_3+k_4}{1-k_2-k_4-k_5}\}$,then ϕ, k satisfy the definition 1.5, therefore $\phi\in\psi$.

Lemma 1.1[6] Let (X, d) be a complex valued metric space. Let $d = d_1 + id_2$,which means:

$d_1 = \mathrm{Re}(d), d_2 = \mathrm{Im}(d)$, $d_1, d_2 : X\times X \to ¡$, then we have the following conclusions:

(i) $|d| = (d_1^2 + d_2^2)^{1/2} : X \times X \to \mathbb{R}$ is a metric on X.

(ii) Let $\{x_n\}$ be a sequence on X, $x \in X$, $x_n \xrightarrow{d} x$ holds if and only if $x_n \xrightarrow{|d|} x$.

(iii) (X, d) is complete if and only if $(X, |d|)$ is complete.

The following two lemmas are the specific expression of lemma 1.1

Lemma 1.2 [4] Let $\{x_n\}$ be a sequence in the complex valued metric space (X, d), $\{x_n\}$ is a Cauchy sequence equivalent to $|d(x_n, x_m)| \to 0$ $(n \to \infty, m > n)$.

Lemma 1.3 [4] Let $\{x_n\}$ be a sequence in the complex valued metric space (X, d), if $\{x_n\}$ is convergent, then the limit point is unique.

The following lemma is easy but very useful in the research of fixed point theory.

Lemma 1.4 [8] (Cauchy Principle) Let $\{x_n\}$ be a sequence in the complex valued metric space (X, d), if there is $0 \le h < 1$ such that $d(x_{n+1}, x_n) \precsim h d(x_n, x_{n-1})$ (for all $n \in \mathbb{N}$)then $\{x_n\}$ is a Cauchy sequence.

II $\varphi-$ Contractive Maps

Theorem 1 Let (X, d) be a complex valued metric space, for all $x \in X$, $D_T(x,x) < +\infty$. Let map: $T : X \to X$, if $|d(Tx, Ty)| \le \varphi(D_T(x, y))$, where φ satisfying definition 1.4, then T has a unique common fixed point in X.

Proof:

Firstly, we prove the existence of fixed point.

Let $x_0 \in X$. We can construct the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0,1,2,...$ For each $m > n$, We let:

$$D_n = D_T(x_n, x_m) = \sup\{|\, d(u,v) \,| : u, v \in \{x_n, x_{n+1}, ..., x_m, x_{m+1}, ...\}\} \tag{1}$$

From (1) we can easily get:

$$D_n = D_T(x_n, x_m) = D_T(x_n, x_n) \tag{2}$$

Since $\{x_{n+1}, x_{n+2}, ...\} \subseteq \{x_n, x_{n+1}, ...\}$, the sequence $\{D_n\}$ is non-increasing.

Meanwhile, from (1), we can get $D_n = D_T(x_n, x_n) \ge 0$. Hence, $\lim_{n\to\infty} D_n = \varepsilon \ge 0$.

If $\lim_{n\to\infty} D_n = \varepsilon > 0$, for every $0 < \varepsilon' < \varepsilon$, there is $\delta = \varepsilon - \varepsilon' + 1 > 0$, from the property of limitation, we can get:

There is $n_0 \in ¥$, when $n \geq n_0$, $\varepsilon' < D_n < \varepsilon' + \delta$, that is:

$$\varepsilon' < D_T(x_n, x_m) < \varepsilon' + \delta \tag{3}$$

Then we can get:

$$\varphi(D_T(x_n, x_m)) \leq \varepsilon' \tag{4}$$

Hence, for each $m > n > n_0$, we have:

$$|d(x_{n+1}, x_{m+1})| = |d(Tx_m, Tx_n)| \leq \varphi(D_T(x_n, x_m)) \leq \varepsilon' \tag{5}$$

Since ε' is arbitrary and from lemma 2, we can get: $\{x_n\}$ is a Cauchy sequence in X.

Hence, for every $m > n$, $\lim_{n\to\infty} d(x_n, x_m) = 0$.

Therefore, $\lim_{n\to\infty} D_n = \lim_{n\to\infty} \sup\{|d(u,v)| : u, v \in \{x_n, x_{n+1}, \ldots\}\} = 0 < \varepsilon$,which is a contradiction.

Then, we have $\lim_{n\to\infty} D_n = 0$.

From $\lim_{n\to\infty} D_n = 0$, we know that:

For every $\varepsilon'' > 0$, there is $n_0 \in ¥$, when $n \geq n_0$, $D_n = D_T(x_n, x_n) < \varepsilon''$. Hence, we can get:

$$|d(x_{n+1}, x_{m+1})| = |d(Tx_n, Tx_m)| \leq \varphi(D_T(x_n, x_m)) < D_T(x_n, x_m) = D_n < \varepsilon'' \tag{6}$$

Since ε'' is arbitrary and from lemma 2, we can get: $\{x_n\}$ is a Cauchy sequence in X.

Since (X, d) is complete, we have:

$$\lim_{n\to\infty} x_n = z \in X .$$

From the proof above, we have the following conclusion:

Let $y_n = T^n z$, for all $n \in ¥$,we have:

$$\lim_{n\to\infty} D_T(T^n z, T^n z) = 0 .$$

Then $|d(x_{n+1}, y_{n+1})| = |d(Tx_n, Ty_n)| \leq \varphi(D_T(x_n, y_n))$.

Since $D_T(x_n, y_n)$ is non-increasing, we can easily get $\lim_{n\to\infty} |d(x_{n+1}, y_{n+1})| = 0$ holds through the proof above. Meanwhile, we have $\lim_{n\to\infty} x_n = z$. Hence, we can know that: $\lim_{n\to\infty} y_n = z$.

By definition 1.1 and lemma 1.1, where $\{T^n z\} \xrightarrow{|d|} z$, then we have:

$$\lim_{n\to\infty} D_T(T^n x_0, T^n z) = 0 \, , D_T(Tz,Tz) = D_T(z,z) \, . \tag{7}$$

In fact, for each $x, y \in X$, we can prove $\lim_{n\to\infty} D_T(T^n x, T^n y) = 0$ holds through the proof above.

Next, we will prove $D_T(Tz,Tz) = D_T(z,z) = 0$.

If $D_T(Tz,Tz) = D_T(z,z)=\varepsilon > 0$, and $\lim_{n\to\infty} D_T(T^n z, T^n z) = 0$ holds， we can lei：

$$\varepsilon = D_T(z,z)=D_T(Tz,Tz) = ... = D_T(T^{n_0} z, T^{n_0} z) > D_T(T^{n_0+1} z, T^{n_0+1} z)$$

Since $D_T(T^{n_0} z, T^{n_0} z) \neq D_T(T^{n_0+1} z, T^{n_0+1} z)$, we can get:

$$D_T(T^{n_0} z, T^{n_0} z) = \sup\{|d(T^{n_0} z, T^n z)|, n > n_0\} \tag{8}$$

However, $d(T^{n_0} z, T^n z) \leq \varphi(D_T(T^{n_0-1} z, T^{n-1} z)) = \varphi(D_T(T^{n_0-1} z, T^{n_0-1} z)) = \varphi(\varepsilon) < \varepsilon$.

Since n is arbitrary, we can get: $\varepsilon = D_T(T^{n_0} z, T^{n_0} z) \leq \varphi(\varepsilon) < \varepsilon$,which is a contraction.

Hence, we can get:

$$D_T(z,z) = D_T(Tz,Tz) = 0 \tag{9}$$

That is: $z = Tz$, which means z is the fixed point of map T .

Secondly， we will prove the fixed point we get is unique.

If there is $u \in X$, such that $Tu = u$ holds. We have：

$$|d(u,z)| = |d(T^{n+1} u, T^{n+1} z)| \leq \varphi(D_T(T^n u, T^n z)) < D_T(T^n u, T^n z) \tag{10}$$

Let $n \to \infty$, we can get：

$|d(u,z)| \leq 0$, that is $|d(u,z)| = 0$ .Then $u = z$ is the unique fixed point for maps T .

From theorem1 and definition 1.4， we can give the linear contraction fixed point theorem:

Corollary 1 Let $(X，d)$ be a complex valued metric space，for every $x \in X$, $D_T(x,x) < +\infty$. Let $T: X \to X$, if $|d(Tx,Ty)| \leq h \cdot |D_T(x,y)|$, $h \in [0,1)$, then the map T has a unique fixed point in X .

Proof： Let $\varphi(t) = h \cdot t$, then we should prove $\varphi(t) = h \cdot t$ satisfying the definition 1.4.

For each $\varepsilon > 0$, there is $\delta = \dfrac{\varepsilon(1-h)}{h} > 0$, when $\varepsilon < t < \varepsilon + \delta$, $\varphi(t) = h \cdot t < h(\varepsilon + \delta) = \varepsilon$. Then, we can easily check $\varphi(t) = h \cdot t$ satisfying the definition 1.4, and corollary 1 satisfying the condition of theorem 1. Obviously, the corollary 1 holds.

In order to partly generalize the theorem 1 into the family of maps, we research a similar control function ϕ as we defined in definition 1.5, the results we obtained are as follows.

III $\phi-$ Contractive Self-Maps

In order to construct the fixed point theorem under this condition, firstly, we generalize the Cauchy principle in lemma 1.4 into a more normal condition.

Theorem 2 Let (X, d) be a complex valued metric space, $\{x_n\}$ is a sequence on X, if there is $h \in \pounds, |h| < 1$,such that $d(x_n, x_{n+1})$ ˆ $h \cdot d(x_{n-1}, x_n)$, for all $n \in \yen$, then $\{x_n\}$ is a Cauchy sequence on X.

Remark 2: Compared with lemma 1.4, we generalize the range of h into complex numbers.

Proof: From the condition, we know:

$$|d(x_n, x_{n+1})| \le |h||d(x_{n-1}, x_n)| \le \ldots \le |h|^{n-1}|d(x_1, x_2)|, \text{ for all } n \in \yen$$

Therefore:

$$\begin{aligned} |d(x_n, x_m)| &\le |d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m})| \\ &\le |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \ldots + |d(x_{n+m-1}, x_{n+m})| \\ &\le (|h|^{n-1} + |h|^{n} + \ldots + |h|^{n+m-2})|d(x_1, x_2)| \end{aligned}$$

Let $n \to \infty$, we can get:

$$\lim_{n\to\infty} |d(x_n, x_m)| \le \lim_{n\to\infty} \frac{|h|^{n-1}}{1-|h|}|d(x_1, x_2)| = 0$$

From lemma 2, we know that $\{x_n\}$ is a Cauchy sequence on X.

Remark 3: if z ˆ $k \cdot z$ $(k, z \in \pounds)$,and $|k| < 1$,then $|z| \le |kz| = |k||z| \le |k|^2|z| \le \ldots \le |k|^n|z|$, let $n \to \infty$ we can get: $|z| = 0$,which means: $z = 0$ holds.

Theorem 3 Let (X, d) be a complex valued metric space, $\{T_i\}_{i=1}^{\infty}$ is a self-map on X. Let nonnegative integer sequence $\{m_i\}_{i=1}^{\infty}$ and $\phi \in \psi$, such that for every $i, j, i \ne j$ and every $x, y \in X$,the following condition holds:

$$d(T_i^{m_i}x, T_j^{m_j}y) \leq \phi(d(x,y), d(x,T_i^{m_i}x), d(y,T_j^{m_j}y), d(x,T_j^{m_j}y), d(y,T_i^{m_i}x))$$

Then, the self-map $\{T_i\}_{i=1}^{\infty}$ has a unique common fixed point on X.

Proof: let $g_i = T_i^{m_i}, i = 1, 2, ...$,for every $i, j, i \neq j$, there is:

$$d(g_i(x), g_j(y)) \leq \phi(d(x,y), d(x, g_i(x)), d(y, g_j(y)), d(x, g_j(y)), d(y, g_i(x))). \tag{11}$$

For each $x_0 \in X$, defined sequence $\{x_n\}$ as follows:

$$x_n = g_n(x_{n-1}), n = 1, 2, ...$$

Then, the (11) turn to:

$$d(x_n, x_{n+1}) = d(g_n(x_{n-1}), g_{n+1}(x_n)) \leq$$
$$\phi(d(x_{n-1}, x_n), d(x_{n-1}, g_n(x_{n-1})), d(x_n, g_{n+1}(x_n)), d(x_{n-1}, g_{n+1}(x_n)), d(x_n, g_n(x_{n-1}))),$$

Calculate and simplify:

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0), \tag{12}$$

From the rectangle inequality, we can get:

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}), \tag{13}$$

Since ϕ is increasing for its fourth and fifth variables, the following condition holds:

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), d(x_n, x_{n+1})), \tag{14}$$

From the property of ϕ, we can get:

$$d(x_n, x_{n+1}) \leq k \cdot d(x_{n-1}, x_n)$$

From lemma 1.4 and theorem2(Cauchy principle), we can get: $\{x_n\}$ is a Cauchy sequence on X. Since X is complete, we know:

$$\lim_{n\to\infty} x_n = z \in X$$

For every $m \in \mathbb{N}$,consider the following condition:

$$d(x_{n+1}, g_m(z)) = d(g_{n+1}(x_n), g_m(z)) \leq \phi(d(x_n, z), d(z, g_m(z)), d(x_n, x_{n+1}), d(z, x_{n+1}), d(x_n, g_m(z)))$$

Let $n \to \infty$,we can get:

$$d(z, g_m(z)) = d(g_{n+1}(x_n), g_m(z)) \leq \phi(0, d(z, g_m(z)), 0, 0, d(z, g_m(z))), \tag{15}$$

Let $y = d(z, g_m(z)), x = 0$, from the property of ϕ, we can get:

$y \leq kx = 0$. Meanwhile, $0 \leq y = d(z, g_m(z))$.

Hence, $d(z,g_m(z))=0$ for every $m\in\mathbb{N}$,that is $z=g_m(z), m=1,2,...$

Therefore, z is a common fixed point for self-map $\{g_m\}_{m=1}^{\infty}$.

Suppose that there is $v\in X$, such that $v=g_m(v)$, for all $m\in\mathbb{N}$, consider the following condition:

$$d(z,v)=d(g_i z,g_j v)\precsim \phi(d(z,v),d(z,v),d(z,v),d(z,v),d(z,v)), \tag{16}$$

We can let $x=y=d(z,v)$, since ϕ is increasing for the fourth variable, we know that:

$$d(z,v)\precsim \phi(d(z,v),d(z,v),d(z,v),d(z,v)+d(z,v),d(z,v)) \tag{17}$$

Therefore, $d(z,v)\precsim k\cdot d(z,v), k\in\mathbb{C}, |k|<1$,from remark 3, we can get: $d(z,v)=0$.

Hence, z is the unique common fixed point for self-map $g_i=T_i^{m_i}, i=1,2,......$.

However, for each $i\in\mathbb{N}$,the following holds:

$$T_i z=T_i g_i z=T_i T_i^{m_i} z=T_i^{m_i} T_i z \tag{18}$$

From this, we can see $T_i z$ is the common fixed point for self-map $\{T_i^{m_i}\}_{i=1}^{\infty}$, since z is the unique common fixed point, we can get:

$$T_i z=z, i=1,2,... \tag{19}$$

Then, z is the common fixed point for self-map $\{T_i\}_{i=1}^{\infty}$, if there is $u\in X$, such that $u=T_i u$, for every $i\in\mathbb{N}$, put it into the contractive condition and simplify, from the property of ϕ , we can get:

$$d(z,u)\precsim k\cdot d(z,u), k\in\mathbb{C}, |k|<1$$

Hence, we can easily check: $d(z,u)=0$, which means z is the unique common fixed point for self-map $\{T_i\}_{i=1}^{\infty}$.

Remark 4: We can't get the conclusion: $\phi(0,0,0,0,0)=0$ by definition 1.4

From theorem 3 and example1.3, we can get the following linear contractive fixed point theorem:

Corollary 2 Let (X,d) be a complex valued metric space, and $T:X\to X$ is a self-map on X. For $k_i, i=1..5$, and $\sum_{i=1}^{5}k_i<1-k_4$,the following condition holds:

$$d(Tx,Ty)\precsim k_1 d(x,y)+k_2 d(x,Tx)+k_3 d(y,Ty)+k_4 d(x,Ty)+k_5 d(y,Tx)$$

Then T has a unique fixed point on X.

Proof: From theorem 3 and example 1.3 we can easily check.

Now, we will give an example of the theorem we proved, in order to calculate more convenient, we only give the example of corollary 2.

Example 3.1 Considering the complex metric space in example 1.2, and defined: $T: X \to X$, such that: $Ta = Tc = a, Tb = c.$ Let $k_1 = 0.3, k_3 = 0.65, k_2 = k_4 = k_5 = 0$, then:

$k_1 + k_2 + k_3 + 2k_4 + k_5 = 0.95 < 1$, and:

$$d(Ta,Tb) = d(a,c) = 3+4\mathrm{i}\ \hat{}\ \ 0.3\times(2+3\mathrm{i}) + 0.65\times(4+5\mathrm{i}) = 3.2+4.15\mathrm{i};$$
$$d(Tb,Tc) = d(a,c) = 3+4\mathrm{i}\ \hat{}\ \ 0.3\times(4+5\mathrm{i}) + 0.65\times(3+4\mathrm{i}) = 3.15+4.1\mathrm{i};$$

Therefore, the self-map T satisfying the contractive condition, from theorem 3 and corollary 2, we know that map T has the unique fixed point a.

IV Conclusion

In this paper, we generalize the main results in [] into complex valued metric space and get some useful corollary. Meanwhile, we defined a new control function and give a new contractive condition. Through generalizing the Cauchy principle in complex valued metric space, we prove the self-maps satisfying the contractions we defined has a unique common fixed point, the results we obtained generalize many conclusion in the complex valued metric space.

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