A STUDY ON THE GEOMETRIC MEANING OF THE DERIVATIVE FROM THE HPM PERSPECTIVE BASED ON THE "DUAL-CORRESPONDENCE PRINCIPLE"

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ABSTRACT

The "General High School Mathematics Curriculum Standards (2017 Edition, Revised in 2020)" advocate that teachers "promote student learning through diverse teaching methods and enhance classroom effectiveness through precise and efficient teaching strategies." Derivatives are a core concept in high school mathematics and a key focus and challenge in teaching. The concept of limits forms the foundation of derivatives. Neglecting to explore the concept of limits within derivatives is one of the reasons for difficulties in future higher mathematics learning. Fredenthal proposed the concept of mathematical recreation, arguing that understanding the historical context of related concepts helps teachers deeply grasp the concepts and design effective teaching strategies. Professor Tu Rongbao's "dual-correspondence principle" of mathematics education emphasizes the correspondence between mathematical knowledge, student cognition, and disciplinary knowledge, which is beneficial for guiding mathematics education research and improving classroom teaching effectiveness. This study adopts a History and Philosophy of Mathematics (HPM) perspective, following two main threads: "correspondence between teaching and learning" and "correspondence between teaching and mathematics." drawing on the mathematical history of Liu Hui's method of inscribing circles and the development of tangents, to understand the transition from the static to the dynamic definition of tangents, achieving a leap from the finite to the infinite. It explores the correspondence of the "dual-correspondence principle" in the teaching design of the geometric meaning of derivatives, and discusses the unit teaching design of the geometric meaning of derivatives, with the aim of enhancing secondary mathematics teaching focused on improving subject-specific core competencies and providing new insights for teaching practice.

Keywords: The geometric meaning of derivatives; the "dual-correspondence principle" of mathematics teaching; HPM.

RAISING THE ISSUE

The "General High School Mathematics Curriculum Standards (2017 Edition, Revised in 2020)" (hereinafter referred to as the "New Curriculum Standards") states that mathematics courses should reveal the cultural connotations of mathematics, emphasizing that mathematics is not only a tool for calculation and reasoning, but also a carrier of ideas and culture, and an important part of human civilization. However, current high school mathematics teaching places too much emphasis on problem-solving skills training and neglects mathematical thinking. As an important chapter in high school mathematics, many students' understanding of derivatives is limited to calculating derivatives, determining monotonic intervals, extrema, and maximum/minimum values, while neglecting the underlying mathematical ideas behind derivatives. Mathematical education has both educational and disciplinary characteristics. Based on this, instructional design should adhere to the "dual-correspondence Principle." This study aims to explore a teaching design for the geometric meaning of derivatives based on the "dual-correspondence Principle" from an HPM perspective, providing new insights for improving the quality of mathematics education.

THEORETICAL BASIS

The dual-correspondence principles of mathematics education include the principle of correspondence between teaching and learning and the principle of correspondence between teaching and mathematics.

Correspondence Between Teaching and Learning

The curriculum standards are uniform, but different students have varying levels of prior learning experience and cognitive development, leading to differing degrees of acceptance of abstract concepts. The principle of correspondence between teaching and learning requires that instruction be adapted to students' cognitive characteristics and developmental levels. Specifically, teachers must design instructional objectives, select teaching materials, and choose teaching methods based on students' cognitive levels and subject-specific foundations at their current educational stage, ensuring that "teachers' instruction" and "students' learning" achieve true unity. Students have already studied the positional relationship between lines and circles and are familiar with the concept of tangents, but most students' cognitive levels are primarily limited to "a line that has exactly one common point with a curve" or "a line on the same side of a curve is a tangent." Teachers should clarify the connection and distinction between tangents of circles and tangents in derivatives to prevent students from confusing the concepts and dampening their interest in learning. Additionally, teachers can expand on the background and applications of derivatives, strengthen exercises linking mathematics to the real world, and help students appreciate the practicality of mathematics, thereby stimulating their interest in learning.

Teaching and Mathematical Correspondence

Teaching and mathematical correspondence involve understanding mathematics education from the perspective of the mathematical discipline, emphasizing its unique abstractness, logicality, and humanistic nature, and valuing its disciplinary value and methodological approaches. In their book *What Is Mathematics?*, Courant and Robbins noted, "The traditional status of mathematics education is now facing severe challenges. Mathematical training sometimes devolves into mechanical, hollow problem-solving exercises, and mathematical research has developed a tendency toward excessive specialization and an overemphasis on abstraction, thereby neglecting the practical value of mathematics." Mathematics education should emphasize the essence of mathematics, not only requiring students to be able to prove and calculate, but more importantly, requiring them to grasp the underlying ideas. The tendency to "de-mathematize" is undesirable. The new curriculum standards for high school calculus require a focus on the historical context, fundamental ideas, and widespread applications of calculus, understanding the concept of limits, de-emphasizing technical calculations, and highlighting geometric intuitive thinking. Every mathematical concept has its historical context. The definition of a tangent has evolved from static to dynamic. Teachers can incorporate the historical evolution of tangents into their lessons, using Liu Hui's "Method of Exhaustion" to explain the "replacing curves with lines" concept, thereby helping students understand the geometric essence of derivatives.

TEACHING DESIGN CASE STUDY

Topic Introduction and Knowledge Review

Scenario Question 1: [Shell Flight Direction Problem] In military films, we often see scenes of artillery firing shells. If we want to determine the direction of a shell's flight at a specific moment in time for precise targeting or ballistic analysis, what should we do?

Scenario Question 2: [Arch Bridge Slope Problem] We can easily calculate the slope of a ramp. Please think about how to calculate the slope of an arch bridge.

Design Intent: Exploring historical origins, this lesson draws inspiration from two major problems closely related to tangent line research in the 17th century—the direction of velocity in curved motion and the slope of an arch bridge. This aims to spark students' interest, help them understand the connection between mathematics and real life, and motivate their learning.

Reviewing the concept of tangents to circles

Scenario Question 3: We have previously studied tangents to circles. Please recall how the definition of a tangent to a circle is stated. Think about whether this definition is applicable. Can it be used to define tangents to other conic sections? Is $y=1$ the tangent of $y=\cos x$? Is $y=0$ the tangent of $y=x^3$? Please explain your reasoning. If not, how should we redefine the tangent?

Design intent: In the past, the definition of a tangent was limited to a static geometric definition, but case studies have shown that this definition does not apply to all curves, leading to cognitive conflicts among students and prompting them to actively seek new definitions of tangents. Teaching should be based on students' existing cognitive levels, and teaching content should be adjusted accordingly.

Exploring a New Definition of Tangents Using the Method of Exhaustion

Exploration 1: Demonstrating the Method of Exhaustion Using Modern Educational Technologies Such as Geogebra

Let us return to the familiar concepts of circles and tangents. Regarding the value of pi, its exact value remains elusive to this day. Ancient Chinese mathematician Liu Hui proposed the "Circle-Cutting Method," which involves constructing a regular polygon inside a circle and continuously increasing the number of sides to approximate the area of the circle. Connecting two adjacent vertices of the regular polygon inscribed in the circle, the line segment connecting these two vertices is referred to as a secant of the circle. Treating one of these points P as a fixed point, and the other point Q as a moving point. As the number of sides of the regular polygon increases, the moving point Q gradually approaches the fixed point P, and the secant line becomes increasingly close to the tangent line. When the number of sides of the regular polygon approaches infinity, the moving point Q infinitely approaches the fixed point P, and the secant line infinitely approaches the tangent line. Thus, we can conclude that the tangent line of a circle is the limit of the secant line between any two points on the circle.

Teachers can demonstrate Liu Hui's method of inscribing a circle dynamically on multimedia devices, allowing students to intuitively understand the relationship between secants and tangents.

Design Intent: Using the tangent to a circle as a starting point, we redefine the concept from a limit perspective to avoid confusion among students. By using graphical tools to visually demonstrate the dynamic relationship between secants and tangents, students can understand how secants approach tangents and grasp the concept of limits. From the perspective of the correspondence between teaching and learning, the art of cutting circles helps students enhance their sense of pride and identity with traditional culture while fostering innovative thinking.

Exploring the Relationship Between Tangents and Derivatives Through the Integration of Numbers and Shapes

Exploration 2: P is a fixed point on curve $f(x)$, and $Q_i(i=1,2,3...)$ is a moving point on curve $f(x)$.

Thought 1: Write the average rate of change of $f(x)$ in the interval from x_0 to x_1; what is its geometric meaning?

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This represents the slope of line PQ_1, which is a tangent line to $f(x)$.

Thought 2: When point Q_i approaches point P along the curve, what changes occur in the secant PQ_i?
(1) Change in slope: The slope of the secant line PQ_i gradually approaches the slope of the tangent line at point P.
(2) Change in position: The secant line PQ_i gradually approaches the position of the tangent line at point P.
(3) Change in trend: The secant line PQ_i approaches point P from different directions and eventually connects seamlessly with the curve at point P.

Thought 3: When point Q_i approaches point P infinitely along the curve, what does line PQ_i ultimately become?
Line PQ_i ultimately becomes the tangent line PT at point P.

Thought 4: What is the relationship between the slope of the secant line PQ_i and the slope k of the tangent line PT?
As point Q_i gradually approaches point P along the curve, the slope of the secant line PQ_i gradually approaches the slope of the tangent line PT. The slope of the tangent line PT is the limit value of the slope of the secant line PQ_i.

Thought 5: Based on the above questions, can you summarize the geometric meaning of the derivative $f^{'}(x_0)$?
For the function $y = f(x)$, if the x-coordinate of point P is x_0, then the x-coordinate of point Q can be expressed as $x_0 + \Delta x$ (where Δx is the increment). At this point, the slope of the secant line PQ is

$$k_{PQ} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The derivative of $f(x)$ at $x = x_0$ is $f^{'}(x_0)$, which is the slope k_{PT} of the tangent line PT.

$$k_{PT} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f^{'}(x_0)$$

Thought 6: What are the similarities and differences between the tangent line of a curve at a point and the tangent line of a circle that you have learned about before?

Similarities:
(1) Reflects the essence of a tangent line: indicates the instantaneous direction of change of the curve at a point.
(2) Consistent geometric meaning: The slope of the tangent line reflects the trend of change of the curve at that point.

Differences:
(1) Differences in definition: The tangent line of a circle is defined as a straight line that shares only one point with the circle and is perpendicular to the radius at the point of tangency; the tangent line

of a curve is defined as the limit position of the two intersection points of the secant line as they approach that point, which is the geometric representation of the derivative.
(2) Differences in scope of application: The definition of a tangent line to a circle is specific to circles as a particular type of curve; the definition of a tangent line to a curve applies to all differentiable function curves, including circles.
(3) Differences in properties: The tangent line to a circle is always perpendicular to the radius at the point of tangency, with its direction determined by the position of the radius; the slope of a tangent line to a general curve is determined by the derivative, with its direction depending on the shape of the curve and its rate of change.

Teacher summary: Through exploration, we have linked the symbolic representation of the derivative to the slope of the tangent line. This connection allows us to understand the instantaneous rate of change of a function from a geometric perspective, while also enabling us to use algebraic methods for precise calculation and analysis, thereby building a bridge between algebra and geometry for further study of functions.

Design Intent: Applying the dual-correspondence Principle of mathematics education, we integrate the historical evolution of tangents into the classroom to guide students in understanding that exploring the essence of things is a progressive process of deepening understanding, distinguishing truth from falsehood, and moving from the superficial to the profound. Through peer-to-peer interaction and teacher-student collaboration, students undergo a complete exploration of the geometric meaning of derivatives, deepening their understanding of the integration of numbers and shapes. This enables students to elevate their intuitive understanding to rational thinking, abstracting and generalizing the geometric meaning of derivatives, with mathematical knowledge emerging naturally.

Deepening the mathematical concept of "replacing curves with straight lines"
Show a picture of a circular flower bed. The circular flower bed is actually made of square bricks, which is a visual representation of Liu Hui's method of inscribing a circle. Each brick (straight line) can be approximated as a segment of a circular arc. Similarly, we can use the tangent line at a point on a curve to approximate the curve near that point. This is an important concept in calculus—the idea of "replacing curves with straight lines." In fact, we are already familiar with this method of using simple objects to approximate complex ones, such as using the rational number 3.1415926 to approximate the irrational number for computational purposes.

Design Intent: Use real-life examples and familiar concepts to reduce students' cognitive load and avoid making them feel intimidated by calculus.

Return to Context: Classic Example Practice
Example 1: Given curve $y = x^2 - 1$, find the slope of the tangent line at point $P(1,0)$.

Example 2: Suppose that the trajectory of a cannonball is described by equation $y = -x^2 + 10x$, where x is the horizontal distance (in kilometers) and y is the height (in kilometers). Find the direction of flight of the cannonball at a horizontal distance of $x = 3$ kilometers.

Design Intent: Example 1 is designed to help students understand the geometric meaning of derivatives and use it to find the slope of a tangent line. Example 2 is designed to help students understand and apply the geometric meaning of derivatives in real-world contexts.

Class Summary
What did you learn in this lesson?
(1) The concept of a tangent line to a curve, the slope of a tangent line, and the geometric meaning of derivatives;
(2) An important methodological approach in calculus—replacing curves with straight lines;
(3) Using derivatives to find the slope of the tangent line at a point on a curve.

Design Intent: Guide students to systematically organize the knowledge from this lesson, reinforce memory, integrate the knowledge system, and develop mathematical thinking and application skills. Additionally, through reflecting on the gains and losses of the learning process, cultivate their ability for self-summary and self-reflection, and provide reference for teachers to improve teaching.

CONCLUSIONS

Mathematics is a product of history. The teaching design based on the “dual-correspondence Principle” of applied mathematics education from the HPM perspective organically integrates teachers' teaching and students' learning. It takes into account the cognitive development patterns of students' learning, enabling them to engage in meaningful learning, while also revealing the disciplinary characteristics and methodological approaches of mathematics during the teaching process. This approach aims to develop students' core mathematical literacy, guiding them to become individuals who can observe the real world using mathematical methods, think about the real world using mathematical reasoning, and express the real world using mathematical language. Implementing the “Dual-correspondence Principle” teaching approach in practice is no easy task. The key lies in enhancing the overall quality of the teaching staff. Teachers must deeply understand and apply theories and methods from educational psychology, pedagogy, and teaching theory to guide their teaching practices. They must also possess a broader and deeper understanding of the knowledge content, theoretical framework, and mathematical culture of the subject.

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